FHSST Authors

The Free High School Science Texts: Textbooks for High School Students Studying the Sciences Mathematics Grades 10 - 12

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Contents

Chapter 10

Equations and Inequalities - Grade 10

10.1 Strategy for Solving Equations

This chapter is all about solving different types of equations for one or two variables. In general, we want to get the unknown variable alone on the left hand side of the equation with all the constants on the right hand side of the equation. For example, in the equation $x - 1 = 0$, we want to be able to write the equation as $x = 1$.

As we saw in section 2.9 (page 13), an equation is like a set of weighing scales, that must always be balanced. When we solve equations, we need to keep in mind that what is done to one side must be done to the other.

Method: Rearranging Equations

You can add, subtract, multiply or divide both sides of an equation by any number you want, as long as you always do it to both sides.

For example, in the equation $x+5-1 = -6$, we want to get x alone on the left hand side of the equation. This means we need to subtract 5 and add 1 on the left hand side. However, because we need to keep the equation balanced, we also need to subtract 5 and add 1 on the right hand side.

$$\begin{aligned} x+5-1 &= -6 \\ x+5-5-1+1 &= -6-5+1 \\ x+0+0 &= -11+1 \\ x &= -10 \end{aligned}$$

In another example, $\frac{2}{3}x = 8$, we must divide by 2 and multiply by 3 on the left hand side in order to get x alone. However, in order to keep the equation balanced, we must also divide by 2 and multiply by 3 on the right hand side.

$$\begin{aligned} \frac{2}{3}x &= 8 \\ \frac{2}{3}x \div 2 \times 3 &= 8 \div 2 \times 3 \\ \frac{2}{2} \times \frac{3}{3} \times x &= \frac{8 \times 3}{2} \\ 1 \times 1 \times x &= 12 \\ x &= 12 \end{aligned}$$

These are the basic rules to apply when simplifying any equation. In most cases, these rules have to be applied more than once, before we have the unknown variable on the left hand side of the equation.

We are now ready to solve some equations!

Important: The following must also be kept in mind:

1. Division by 0 is undefined.
2. If $\frac{x}{y} = 0$, then $x = 0$ and $y \neq 0$, because division by 0 is undefined.

Activity :: Investigation : Strategy for Solving Equations
In the following, identify what is wrong.

$$\begin{aligned} 4x - 8 &= 3(x-2) \\ 4(x-2) &= 3(x-2) \\ \frac{4(x-2)}{(x-2)} &= \frac{3(x-2)}{(x-2)} \\ 4 &= 3 \end{aligned}$$

10.2 Solving Linear Equations

The simplest equation to solve is a linear equation. A linear equation is an equation where the power on the variable(letter, e.g. x) is 1(one). The following are examples of linear equations.

$$\begin{aligned} 2x + 2 &= 1 \\ \frac{2-x}{3x+1} &= 2 \\ \frac{4}{3}x - 6 &= 7x + 2 \end{aligned}$$

In this section, we will learn how to find the value of the variable that makes both sides of the linear equation true. For example, what value of x makes both sides of the very simple equation, $x + 1 = 1$ true.

Since the highest power on the variable is one(1) in a linear equation, there is at most *one solution* or *root* for the equation.

This section relies on all the methods we have already discussed: multiplying out expressions, grouping terms and factorisation. Make sure that you are comfortable with these methods, before trying out the work in the rest of this chapter.

$$\begin{aligned} 2x + 2 &= 1 \\ 2x &= 1 - 2 \quad \text{(like terms together)} \\ 2x &= -1 \quad \text{(simplified as much a possible)} \end{aligned}$$

Now we see that $2x = -1$. This means if we divide both sides by 2, we will get:

$$x = -\frac{1}{2}$$

If we substitute $x = -\frac{1}{2}$, back into the original equation, we get:

$$\begin{aligned} & 2x + 2 \\ = & \ 2(-\frac{1}{2}) + 2 \\ = & \ -1 + 2 \\ = & \ 1 \end{aligned}$$

That is all that there is to solving linear equations.

Important: Solving Equations

When you have found the solution to an equation, substitute the solution into the original equation, to check your answer.

Method: Solving Equations

The general steps to solve equations are:

1. Expand(Remove) all brackets.
2. "Move" all terms with the variable to the left hand side of equation, and all constant terms (the numbers) to the right hand side of the equal to-sign. Bearing in mind that the sign of the terms will chance(from (+) to (-) or vice versa, as they "cross over" the equal to sign.
3. Group all like terms together and simplify as much as possible.
4. Factorise if necessary.
5. Find the solution.
6. Substitute solution into **original** equation to check answer.

Worked Example 28: Solving Linear Equations

Question: Solve for x: $4 - x = 4$
Answer
Step 1 : Determine what is given and what is required
We are given $4 - x = 4$ and are required to solve for x.
Step 2 : Determine how to approach the problem
Since there are no brackets, we can start with grouping like terms and then simplifying.
Step 3 : Solve the problem

$$\begin{aligned} 4 - x &= 4 \\ -x &= 4 - 4 \quad \text{(move all constant terms (numbers) to the RHS (right hand side))} \\ -x &= 0 \quad \text{(group like terms together)} \\ -x &= 0 \quad \text{(simplify grouped terms)} \\ -x &= 0 \\ \therefore \quad x &= 0 \end{aligned}$$

Step 4 : Check the answer

Substitute solution into original equation:

$$4 - 0 = 4$$

$$4 = 4$$

Since both sides are equal, the answer is correct.
Step 5 : Write the Final Answer
The solution of $4 - x = 4$ is $x = 0$.

Worked Example 29: Solving Linear Equations

Question: Solve for x: $4(2x - 9) - 4x = 4 - 6x$
Answer
Step 1 : Determine what is given and what is required
We are given $4(2x - 9) - 4x = 4 - 6x$ and are required to solve for x.
Step 2 : Determine how to approach the problem
We start with expanding the brackets, then grouping like terms and then simplifying.
Step 3 : Solve the problem

$$\begin{aligned}
4(2x-9)-4x &= 4-6x \\
8x-36-4x &= 4-6x \quad \text{(expand the brackets)} \\
8x-4x+6x &= 4+36 \quad \text{(move all terms with x to the LHS and all constant terms to the RHS of the =)} \\
(8x-4x+6x) &= (4+36) \quad \text{(group like terms together)} \\
10x &= 40 \quad \text{(simplify grouped terms)} \\
\frac{10}{10}x &= \frac{40}{10} \quad \text{(divide both sides by 10)} \\
x &= 4
\end{aligned}$$

Step 4 : Check the answer
Substitute solution into original equation:

$$\begin{aligned}
4(2(4)-9)-4(4) &= 4-6(4) \\
4(8-9)-16 &= 4-24 \\
4(-1)-16 &= -20 \\
-4-16 &= -20 \\
-20 &= -20
\end{aligned}$$

Since both sides are equal to -20, the answer is correct.
Step 5 : Write the Final Answer
The solution of $4(2x - 9) - 4x = 4 - 6x$ is $x = 4$.

Worked Example 30: Solving Linear Equations

Question: Solve for x: $\frac{2-x}{3x+1} = 2$
Answer

Step 1 : Determine what is given and what is required
We are given $\frac{2-x}{3x+1} = 2$ and are required to solve for x.
Step 2 : Determine how to approach the problem
Since there is a denominator of (3x+1), we can start by multiplying both sides of the equation by (3x+1). But because division by 0 is not permissible, there is a restriction on a value for x. ($x \neq \frac{-1}{3}$)
Step 3 : Solve the problem

$$\begin{aligned}
\frac{2-x}{3x+1} &= 2 \\
(2-x) &= 2(3x+1) \\
2-x &= 6x+2 \quad \text{(remove/expand brackets)} \\
-x-6x &= 2-2 \quad \text{(move all terms containing x to the LHS and all constant terms (numbers) to the RHS.} \\
-7x &= 0 \quad \text{(simplify grouped terms)} \\
x &= 0 \div (-7) \\
therefore \quad x &= 0 \quad \text{zero divide by any number is 0}
\end{aligned}$$

Step 4 : Check the answer
Substitute solution into original equation:

$$\begin{aligned}
\frac{2-(0)}{3(0)+1} &= 2 \\
\frac{2}{1} &= 2
\end{aligned}$$

Since both sides are equal to 2, the answer is correct.

Step 5 : Write the Final Answer
The solution of $\frac{2-x}{3x+1} = 2$ is $x = 0$.

Worked Example 31: Solving Linear Equations

Question: Solve for x: $\frac{4}{3}x - 6 = 7x + 2$
Answer
Step 1 : Determine what is given and what is required
We are given $\frac{4}{3}x - 6 = 7x + 2$ and are required to solve for x.
Step 2 : Determine how to approach the problem
We start with multiplying each of the terms in the equation by 3, then grouping like terms and then simplifying.
Step 3 : Solve the problem

$$\begin{aligned}
\frac{4}{3}x - 6 &= 7x+2 \\
4x-18 &= 21x+6 \quad \text{(each term is multiplied by 3} \\
4x-21x &= 6+18 \quad \text{(move all terms with } x \text{ to the LHS and all constant terms to the RHS of the =)} \\
-17x &= 24 \quad \text{(simplify grouped terms)} \\
\frac{-17}{-17}x &= \frac{24}{-17} \quad \text{(divide both sides by -17)} \\
x &= \frac{-24}{17}
\end{aligned}$$

Step 4 : Check the answer
Substitute solution into original equation:

$$\begin{aligned}
\frac{4}{3} \times \frac{-24}{17} - 6 &= 7 \times \frac{-24}{17} + 2 \\
\frac{4 \times (-8)}{(17)} - 6 &= \frac{7 \times (-24)}{17} + 2 \\
\frac{(-32)}{17} - 6 &= \frac{-168}{17} + 2 \\
\frac{-32 - 102}{17} &= \frac{(-168) + 34}{17} \\
\frac{-134}{17} &= \frac{-134}{17}
\end{aligned}$$

Since both sides are equal to $\frac{-134}{17}$, the answer is correct.

Step 5 : Write the Final Answer
The solution of $\frac{4}{3}x - 6 = 7x + 2$ is, $x = \frac{-24}{17}$.

Exercise: Solving Linear Equations

1. Solve for y: $2y - 3 = 7$
2. Solve for w: $-3w = 0$
3. Solve for z: $4z = 16$
4. Solve for t: $12t + 0 = 144$
5. Solve for x: $7 + 5x = 62$
6. Solve for y: $55 = 5y + \frac{3}{4}$
7. Solve for z: $5z = 3z + 45$
8. Solve for a: $23a - 12 = 6 + 2a$
9. Solve for b: $12 - 6b + 34b = 2b - 24 - 64$
10. Solve for c: $6c + 3c = 4 - 5(2c - 3)$.
11. Solve for p: $18 - 2p = p + 9$
12. Solve for q: $\frac{4}{q} = \frac{16}{24}$
13. Solve for q: $\frac{4}{1} = \frac{q}{2}$
14. Solve for r: $-(-16 - r) = 13r - 1$
15. Solve for d: $6d - 2 + 2d = -2 + 4d + 8$
16. Solve for f: $3f - 10 = 10$
17. Solve for v: $3v + 16 = 4v - 10$
18. Solve for k: $10k + 5 + 0 = -2k + -3k + 80$
19. Solve for j: $8(j - 4) = 5(j - 4)$
20. Solve for m: $6 = 6(m + 7) + 5m$

10.3 Solving Quadratic Equations

A quadratic equation is an equation where the power on the variable is at most 2. The following are examples of quadratic equations.

$$\begin{aligned} 2x^2 + 2x &= 1 \\ \frac{2-x}{3x+1} &= 2x \\ \frac{4}{3}x - 6 &= 7x^2 + 2 \end{aligned}$$

Quadratic equations differ from linear equations by the fact that a linear equation only has one solution, while a quadratic equation has *at most* two solutions. There are some special situations when a quadratic equation only has one solution.

We solve quadratic equations by factorisation, that is writing the quadratic as a product of two expressions in brackets. For example, we know that:

$$(x+1)(2x-3) = 2x^2 - x - 3.$$

In order to solve:

$$2x^2 - x - 3 = 0$$

we need to be able to write $2x^2 - x - 3$ as $(x+1)(2x-3)$, which we already know how to do.

Activity :: Investigation : Factorising a Quadratic
Factorise the following quadratic expressions:

1. $x + x^2$
2. $x^2 + 1 + 2x$
3. $x^2 - 4x + 5$
4. $16x^2 - 9$
5. $4x^2 + 4x + 1$

Being able to factorise a quadratic means that you are one step away from solving a quadratic equation. For example, $x^2 - 3x - 2 = 0$ can be written as $(x-1)(x-2) = 0$. This means that both $x - 1 = 0$ and $x - 2 = 0$, which gives $x = 1$ and $x = 2$ as the two solutions to the quadratic equation $x^2 - 3x - 2 = 0$.

Method: Solving Quadratic Equations

1. First divide the entire equation by any common factor of the coefficients, so as to obtain an equation of the form $ax^2 + bx + c = 0$ where a, b and c have no common factors. For example, $2x^2 + 4x + 2 = 0$ can be written as $x^2 + 2x + 1 = 0$ by dividing by 2.
2. Write $ax^2 + bx + c$ in terms of its factors $(rx + s)(ux + v)$.
   This means $(rx + s)(ux + v) = 0$.
3. Once writing the equation in the form $(rx + s)(ux + v) = 0$, it then follows that the two solutions are $x = -\frac{s}{r}$ or $x = -\frac{u}{v}$.

Extension: Solutions of Quadratic Equations
There are two solutions to a quadratic equation, because any **one** of the values can solve the equation.

Worked Example 32: Solving Quadratic Equations

Question: Solve for x: $3x^2 + 2x - 1 = 0$

Answer

Step 1 : Find the factors of $3x^2 + 2x - 1$

As we have seen the factors of $3x^2 + 2x - 1$ are $(x + 1)$ and $(3x - 1)$.

Step 2 : Write the equation with the factors

$$(x + 1)(3x - 1) = 0$$

Step 3 : Determine the two solutions

We have

$$x + 1 = 0$$

or

$$3x - 1 = 0$$

Therefore, $x = -1$ or $x = \frac{1}{3}$.

Step 4 : Write the final answer

$3x^2 + 2x - 1 = 0$ for $x = -1$ or $x = \frac{1}{3}$.

Worked Example 33: Solving Quadratic Equations

Question: Solve for x: $\sqrt{x + 2} = x$

Answer

Step 1 : Square both sides of the equation

Both sides of the equation should be squared to remove the square root sign.

$$x + 2 = x^2$$

Step 2 : Write equation in the form $ax^2 + bx + c = 0$

$$\begin{aligned} x + 2 &= x^2 \quad \text{(subtract } x^2 \text{ to both sides)} \\ x + 2 - x^2 &= 0 \quad \text{(divide both sides by -1)} \\ -x - 2 + x^2 &= 0 \\ x^2 - x + 2 &= 0 \end{aligned}$$

Step 3 : Factorise the quadratic

$$x^2 - x + 2$$

The factors of $x^2 - x + 2$ are $(x - 2)(x + 1)$.

Step 4 : Write the equation with the factors

$$(x - 2)(x + 1) = 0$$

Step 5 : Determine the two solutions

We have

$$x + 1 = 0$$

or

$$x - 2 = 0$$

Therefore, $x = -1$ or $x = 2$.

Step 6 : Check whether solutions are valid
Substitute $x = -1$into the original equation $\sqrt{x+2} = x$:

$$\begin{aligned} LHS &= \sqrt{(-1)+2} \\ &= \sqrt{1} \\ &= 1 \\ but & \\ RHS &= (-1) \end{aligned}$$

Therefore LHS$\neq$RHS
Therefore $x \neq -1$
Now substitute $x = 2$ into original equation $\sqrt{x+2} = x$:

$$\begin{aligned} LHS &= \sqrt{2+2} \\ &= \sqrt{4} \\ &= 2 \\ and & \\ RHS &= 2 \end{aligned}$$

Therefore LHS = RHS
Therefore $x = 2$ is the only valid solution
Step 7 : Write the final answer
$\sqrt{x+2} = x$ for $x = 2$ only.

Worked Example 34: Solving Quadratic Equations

Question: Solve the equation: $x^2 + 3x - 4 = 0$.
Answer
Step 1 : Check if the equation is in the form $ax^2 + bx + c = 0$
The equation is in the required form, with $a = 1$.
Step 2 : Factorise the quadratic
You need the factors of 1 and 4 so that the middle term is $+3$ So the factors are:
$(x-1)(x+4)$
Step 3 : Solve the quadratic equation

$$x^2 + 3x - 4 = (x-1)(x+4) = 0 \tag{10.1}$$

Therefore $x = 1$ or $x = -4$.
Step 4 : Write the final solution
Therefore the solutions are $x = 1$ or $x = -4$.

Worked Example 35: Solving Quadratic Equations

Question: Find the roots of the quadratic equation $0 = -2x^2 + 4x - 2$.
Answer
Step 1 : Determine whether the equation is in the form $ax^2 + bx + c = 0$, with no common factors.
There is a common factor: -2. Therefore, divide both sides of the equation by -2.

$$\begin{aligned} -2x^2 + 4x - 2 &= 0 \\ x^2 - 2x + 1 &= 0 \end{aligned}$$

Step 2 : Factorise $x^2 - 2x + 1$
The middle term is negative. Therefore, the factors are $(x - 1)(x - 1)$
If we multiply out $(x - 1)(x - 1)$, we get $x^2 - 2x + 1$.
Step 3 : Solve the quadratic equation

$$x^2 - 2x + 1 = (x - 1)(x - 1) = 0$$

In this case, the quadratic is a perfect square, so there is only one solution for x: $x = 1$.
Step 4 : Write the final solution
The root of $0 = -2x^2 + 4x - 2$ is $x = 1$.

Exercise: Solving Quadratic Equations

1. Solve for x: $(3x + 2)(3x - 4) = 0$
2. Solve for a: $(5a - 9)(a + 6) = 0$
3. Solve for x: $(2x + 3)(2x - 3) = 0$
4. Solve for x: $(2x + 1)(2x - 9) = 0$
5. Solve for x: $(2x - 3)(2x - 3) = 0$
6. Solve for x: $20x + 25x^2 = 0$
7. Solve for a: $4a^2 - 17a - 77 = 0$
8. Solve for x: $2x^2 - 5x - 12 = 0$
9. Solve for b: $-75b^2 + 290b - 240 = 0$
10. Solve for y: $2y = \frac{1}{3}y^2 - 3y + 14\frac{2}{3}$
11. Solve for θ: $\theta^2 - 4\theta = -4$
12. Solve for q: $-q^2 + 4q - 6 = 4q^2 - 5q + 3$
13. Solve for t: $t^2 = 3t$
14. Solve for w: $3w^2 + 10w - 25 = 0$
15. Solve for v: $v^2 - v + 3$
16. Solve for x: $x^2 - 4x + 4 = 0$
17. Solve for t: $t^2 - 6t = 7$
18. Solve for x: $14x^2 + 5x = 6$
19. Solve for t: $2t^2 - 2t = 12$
20. Solve for y: $3y^2 + 2y - 6 = y^2 - y + 2$

10.4 Exponential Equations of the form $ka^{(x+p)} = m$

examples solved by trial and error)

Exponential equations generally have the unknown variable as the power. The following are examples of exponential equations:

$$
\begin{aligned}
2^x &= 1 \\
\frac{2^{-x}}{3^{x+1}} &= 2 \\
\frac{4}{3} - 6 &= 7^x + 2
\end{aligned}
$$

You should already be familiar with exponential notation. Solving exponential equations are simple, if we remember how to apply the laws of exponentials.

Activity :: Investigation : Solving Exponential Equations
Solve the following equations by completing the table:

$2^x = 2$	x						
	-3	-2	-1	0	1	2	3
2^x							

$3^x = 9$	x						
	-3	-2	-1	0	1	2	3
3^x							

$2^{x+1} = 8$	x						
	-3	-2	-1	0	1	2	3
2^{x+1}							

10.4.1 Algebraic Solution

Definition: Equality for Exponential Functions
If a is a positive number such that $a > 0$, then:

$$a^x = a^y$$

if and only if:

$$x = y$$

.

This means that if we can write all terms in an equation with the same base, we can solve the exponential equations by equating the indices. For example take the equation $3^{x+1} = 9$. This can be written as:

$$3^{x+1} = 3^2.$$

Since the bases are equal (to 3), we know that the exponents must also be equal. Therefore we can write:

$$x + 1 = 2.$$

This gives:

$$x = 1.$$

Method: Solving Exponential Equations

Try to write all terms with the same base.

Activity :: Investigation : Exponential Numbers

Write the following with the same base. The base is the first in the list. For example, in the list 2, 4, 8, the base is two and we can write 4 as 2^2.

1. 2,4,8,16,32,64,128,512,1024
2. 3,9,27,81,243
3. 5,25,125,625
4. 13,169
5. $2x$, $4x^2$, $8x^3$, $49x^8$

Worked Example 36: Solving Exponential Equations

Question: Solve for x: $2^x = 2$

Answer

Step 1 : Try to write all terms with the same base.

All terms are written with the same base.

$$2^x = 2^1$$

Step 2 : Equate the indices

$$x = 1$$

Step 3 : Check your answer

$$\begin{aligned} & 2^x \\ = & 2^{(1)} \\ = & 2^1 \end{aligned}$$

Since both sides are equal, the answer is correct.

Step 4 : Write the final answer

$$x = 1$$

is the solution to $2^x = 2$.

Worked Example 37: Solving Exponential Equations

Question: Solve:

$$2^{x+4} = 4^{2x}$$

Answer

Step 1 : Try to write all terms with the same base.

$$\begin{aligned} 2^{x+4} &= 4^{2x} \\ 2^{x+4} &= 2^{2(2x)} \\ 2^{x+4} &= 2^{4x} \end{aligned}$$

Step 2 : Equate the indices

$$x + 4 = 4x$$

Step 3 : Solve for x

$$\begin{aligned} x + 4 &= 4x \\ x - 4x &= -4 \\ -3x &= -4 \\ x &= \frac{-4}{-3} \\ x &= \frac{4}{3} \end{aligned}$$

Step 4 : Check your answer

$$\begin{aligned} \text{LHS} &= 2^{x+4} \\ &= 2^{(\frac{4}{3}+4)} \\ &= 2^{\frac{16}{3}} \\ &= (2^{16})^{\frac{1}{3}} \\ \text{RHS} &= 4^{2x} \\ &= 4^{2(\frac{4}{3})} \\ &= 4^{\frac{8}{3}} \\ &= (4^8)^{\frac{1}{3}} \\ &= ((2^2)^8)^{\frac{1}{3}} \\ &= (2^{16})^{\frac{1}{3}} \\ &= \text{LHS} \end{aligned}$$

Since both sides are equal, the answer is correct.

Step 5 : Write the final answer

$$x = \frac{4}{3}$$

is the solution to $2^{x+4} = 4^{2x}$.

?

Exercise: Solving Exponential Equations

1. Solve the following exponential equations.

   a. $2^{x+5} = 2^5$ b. $3^{2x+1} = 3^3$ c. $5^{2x+2} = 5^3$

   d. $6^{5-x} = 6^{12}$ e. $64^{x+1} = 16^{2x+5}$ f. $125^x = 5$

2. Solve: $3^{9x-2} = 27$

3. Solve for k: $81^{k+2} = 27^{k+4}$
4. The growth of an algae in a pond is can be modeled by the function $f(t) = 2^t$. Find the value of t such that $f(t) = 128$?
5. Solve for x: $25^{(1-2x)} = 5^4$
6. Solve for x: $27^x \times 9^{x-2} = 1$

10.5 Linear Inequalities

graphically;

Activity :: Investigation : Inequalities on a Number Line
Represent the following on number lines:

1. $x = 4$
2. $x < 4$
3. $x \leq 4$
4. $x \geq 4$
5. $x > 4$

A linear inequality is similar to a linear equation and has the power on the variable is equal to 1. The following are examples of linear inequalities.

$$\begin{aligned} 2x + 2 &\leq 1 \\ \frac{2-x}{3x+1} &\geq 2 \\ \frac{4}{3}x - 6 &< 7x + 2 \end{aligned}$$

The methods used to solve linear inequalities are identical to those used to solve linear equations. The only difference occurs when there is a multiplication or a division that involves a minus sign. For example, we know that $8 > 6$. If both sides of the inequality are divided by -2, -4 is not greater than -3. Therefore, the inequality must switch around, making $-4 < -3$.

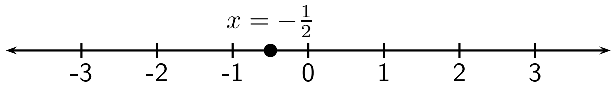
Important: When you divide or multiply both sides of an inequality by any number with a minus sign, the direction of the inequality changes.

For example, if $x < 1$, then $-x > -1$.

In order to compare am inequality to a normal equation, we shall solve an equation first. Solve $2x + 2 = 1$.

$$\begin{aligned} 2x + 2 &= 1 \\ 2x &= 1 - 2 \\ 2x &= -1 \\ x &= -\frac{1}{2} \end{aligned}$$

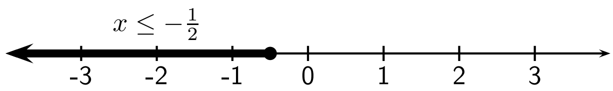
If we represent this answer on a number line, we get



Now let us solve the inequality $2x + 2 \leq 1$.

$$\begin{aligned} 2x + 2 &\leq 1 \\ 2x &\leq 1 - 2 \\ 2x &\leq -1 \\ x &\leq -\frac{1}{2} \end{aligned}$$

If we represent this answer on a number line, we get



As you can see, for the equation, there is only a single value of x for which the equation is true. However, for the inequality, there is a range of values for which the inequality is true. This is the main difference between an equation and an inequality.

Worked Example 38: Linear Inequalities

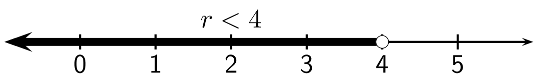
Question: Solve for r: $6 - r > 2$
Answer
Step 1 : Move all constants to the RHS

$$\begin{aligned} -r &> 2 - 6 \\ -r &> -4 \end{aligned}$$

Step 2 : Multiply both sides by -1
When you multiply by a minus sign, the direction of the inequality changes.

$$r < 4$$

Step 3 : Represent answer graphically



Worked Example 39: Linear Inequalities

Question: Solve for q: $4q + 3 < 2(q + 3)$ and represent solution on a number line.
Answer
Step 1 : Expand all brackets

$$\begin{aligned} 4q + 3 &< 2(q + 3) \\ 4q + 3 &< 2q + 6 \end{aligned}$$

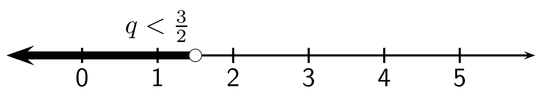
Step 2 : Move all constants to the RHS and all unknowns to the LHS

$$\begin{aligned} 4q + 3 &< 2q + 6 \\ 4q - 2q &< 6 - 3 \\ 2q &< 3 \end{aligned}$$

Step 3 : Solve inequality

$$\begin{aligned} 2q &< 3 \quad \text{Divide both sides by 2} \\ q &< \frac{3}{2} \end{aligned}$$

Step 4 : Represent answer graphically



Worked Example 40: Compound Linear Inequalities

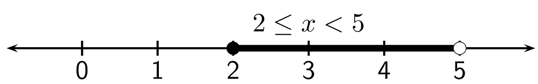
Question: Solve for x: $5 \leq x + 3 < 8$ and represent solution on a number line.

Answer

Step 1 : Subtract 3 from Left, middle and right of inequalities

$$\begin{aligned} 5 - 3 \leq \quad & x + 3 - 3 \quad < 8 - 3 \\ 2 \leq \quad & x \quad < 5 \end{aligned}$$

Step 2 : Represent answer graphically



Exercise: Linear Inequalities

1. Solve for x and represent the solution graphically:
   (a) $3x + 4 > 5x + 8$
   (b) $3(x - 1) - 2 \leq 6x + 4$
   (c) $\frac{x-7}{3} > \frac{2x-3}{2}$
   (d) $-4(x - 1) < x + 2$
   (e) $\frac{1}{2}x + \frac{1}{3}(x - 1) \geq \frac{5}{6}x - \frac{1}{3}$
2. Solve the following inequalities. Illustrate your answer on a number line if x is a real number.
   (a) $-2 \leq x - 1 < 3$

(b) $-5 < 2x - 3 \leq 7$

3. Solve for x: $7(3x + 2) - 5(2x - 3) > 7$.
   Illustrate this answer on a number line.

10.6 Linear Simultaneous Equations

Thus far, all equations that have been encountered have one unknown variable, that must be solved for. When two unknown variables need to be solved for, two equations are required and these equations are known as simultaneous equations. The solutions to the system of simultaneous equations, are the values of the unknown variables which satisfy the system of equations simultaneously, that means at the same time. In general, if there are n unknown variables, then n equations are required to obtain a solution for each of the n variables.

An example of a system of simultaneous equations is:

$$\begin{aligned} 2x + 2y &= 1 \\ \frac{2-x}{3y+1} &= 2 \end{aligned} \tag{10.2}$$

10.6.1 Finding solutions

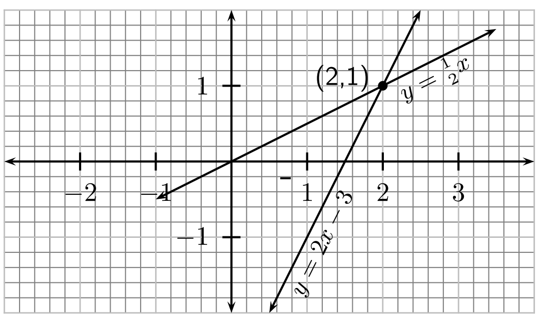
In order to find a numerical value for an unknown variable, one must have at least as many independent equations as variables. We solve simultaneous equations graphically and algebraically/

10.6.2 Graphical Solution

Simultaneous equations can also be solved graphically. If the graphs corresponding to each equation is drawn, then the solution to the system of simultaneous equations is the co-ordinate of the point at which both graphs intersect.

$$\begin{aligned} x &= 2y \\ y &= 2x - 3 \end{aligned} \tag{10.3}$$

Draw the graphs of the two equations in (10.3).



The intersection of the two graphs is (2,1). So the solution to the system of simultaneous equations in (10.3) is $y = 1$ and $x = 2$.

This can be shown algebraically as:

$$\begin{aligned} x &= 2y \\ \therefore \quad y &= 2(2y) - 3 \\ y - 4y &= -3 \\ -3y &= -3 \\ y &= 1 \\ \text{Substitute into the first equation:} \quad x &= 2(1) \\ &= 2 \end{aligned}$$

Worked Example 41: Simultaneous Equations

Question: Solve the following system of simultaneous equations graphically.

$$\begin{aligned} 4y + 3x &= 100 \\ 4y - 19x &= 12 \end{aligned}$$

Answer

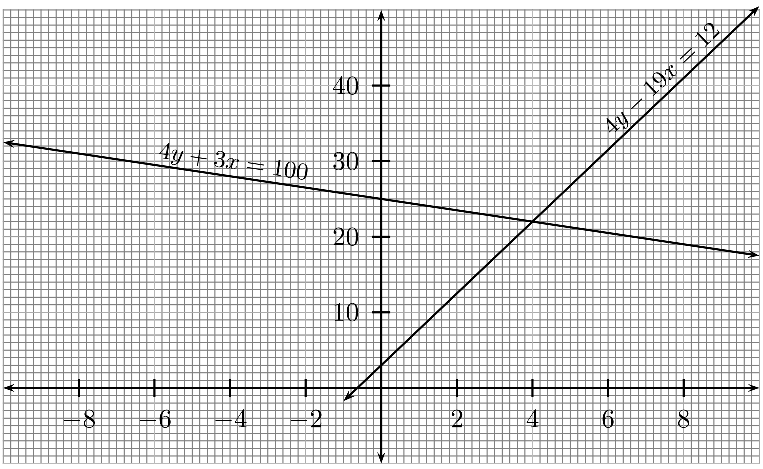
Step 1 : Draw the graphs corresponding to each equation.

For the first equation:

$$\begin{aligned} 4y + 3x &= 100 \\ 4y &= 100 - 3x \\ y &= 25 - \frac{3}{4}x \end{aligned}$$

and for the second equation:

$$\begin{aligned} 4y - 19x &= 12 \\ 4y &= 19x + 12 \\ y &= \frac{19}{4}x + 3 \end{aligned}$$



Step 2 : Find the intersection of the graphs.

The graphs intersect at (4,22).

Step 3 : Write the solution of the system of simultaneous equations as given by the intersection of the graphs.

$$\begin{aligned} x &= 4 \\ y &= 22 \end{aligned}$$

10.6.3 Solution by Substitution

A common algebraic technique is the substitution method: try to solve one of the equations for one of the variables and substitute the result into the other equations, thereby reducing the number of equations and the number of variables by 1. Continue until you reach a single equation with a single variable, which (hopefully) can be solved; back substitution then yields the values for the other variables.

In the example (**??**), we first solve the first equation for x:

$$x = \frac{1}{2} - y$$

and substitute this result into the second equation:

$$\begin{aligned}
\frac{2-x}{3y+1} &= 2 \\
\frac{2-(\frac{1}{2}-y)}{3y+1} &= 2 \\
2-(\frac{1}{2}-y) &= 2(3y+1) \\
2-\frac{1}{2}+y &= 6y+2 \\
y-6y &= -2+\frac{1}{2}+2 \\
-5y &= \frac{1}{2} \\
y &= -\frac{1}{10}
\end{aligned}$$

$$\begin{aligned}
\therefore \quad x &= \frac{1}{2} - y \\
&= \frac{1}{2} - (-\frac{1}{10}) \\
&= \frac{6}{10} \\
&= \frac{3}{5}
\end{aligned}$$

The solution for the system of simultaneous equations (**??**) is:

$$\begin{aligned}
x &= \frac{3}{5} \\
y &= -\frac{1}{10}
\end{aligned}$$

Worked Example 42: Simultaneous Equations

Question: Solve the following system of simultaneous equations:

$$\begin{aligned}
4y + 3x &= 100 \\
4y - 19x &= 12
\end{aligned}$$

Answer

Step 1 : If the question, does not explicitly ask for a graphical solution, then the system of equations should be solved algebraically.

Step 2 : Make x the subject of the first equation.

$$\begin{aligned} 4y + 3x &= 100 \\ 3x &= 100 - 4y \\ x &= \frac{100 - 4y}{3} \end{aligned}$$

Step 3 : Substitute the value obtained for x into the second equation.

$$\begin{aligned} 4y - 19(\frac{100 - 4y}{3}) &= 12 \\ 12y - 19(100 - 4y) &= 36 \\ 12y - 1900 + 76y &= 36 \\ 88y &= 1936 \\ y &= 22 \end{aligned}$$

Step 4 : Substitute into the equation for x.

$$\begin{aligned} x &= \frac{100 - 4(22)}{3} \\ &= \frac{100 - 88}{3} \\ &= \frac{12}{3} \\ &= 4 \end{aligned}$$

Step 5 : Substitute the values for x and y into both equations to check the solution.

$$\begin{aligned} 4(22) + 3(4) = 88 + 12 &= 100 \quad \checkmark \\ 4(22) - 19(4) = 88 - 76 &= 12 \quad \checkmark \end{aligned}$$

Worked Example 43: Bicycles and Tricycles

Question: A shop sells bicycles and tricycles. In total there are 7 cycles and 19 wheels. Determine how many of each there are, if a bicycle has two wheels and a tricycle has three wheels.

Answer

Step 1 : Identify what is required

The number of bicycles and the number of tricycles are required.

Step 2 : Set up the necessary equations

If b is the number of bicycles and t is the number of tricycles, then:

$$\begin{aligned} b + t &= 7 \\ 2b + 3t &= 19 \end{aligned}$$

Step 3 : Solve the system of simultaneous equations using substitution.

$$\begin{aligned}
b &= 7-t \\
\text{Into second equation: } 2(7-t)+3t &= 19 \\
14-2t+3t &= 19 \\
t &= 5 \\
\text{Into first equation: } b &= 7-5 \\
&= 2
\end{aligned}$$

Step 4 : Check solution by substituting into original system of equations.

$$\begin{aligned}
2+5 &= 7 \quad \checkmark \\
2(2)+3(5)=4+15 &= 19 \quad \checkmark
\end{aligned}$$

?

Exercise: Simultaneous Equations

1. Solve graphically and confirm your answer algebraically: $3a-2b7=0$, $a-4b+1=0$
2. Solve algebraically: $15c+11d-132=0$, $2c+3d-59=0$
3. Solve algebraically: $-18e-18+3f=0$, $e-4f+47=0$
4. Solve graphically: $x+2y=7$, $x+y=0$

10.7 Mathematical Models

10.7.1 Introduction

Tom and Jane are friends. Tom picked up Jane's Physics test paper, but will not tell Jane what her marks are. He knows that Jane hates maths so he decided to tease her. Tom says: "I have 2 marks more than you do and the sum of both our marks is equal to 14. How much did we get?"

Let's help Jane find out what her marks are. We have two unknowns, Tom's mark (which we shall call t) and Jane's mark (which we shall call j). Tom has 2 more marks than Jane. Therefore,

$$t=j+2$$

Also, both marks add up to 14. Therefore,

$$t+j=14$$

The two equations make up a set of linear (because the highest power is one) simultaneous equations, which we know how to solve! Substitute for t in the second equation to get:

$$\begin{aligned} t+j &= 14 \\ j+2+j &= 14 \\ 2j+2 &= 14 \\ 2(j+1) &= 14 \\ j+1 &= 7 \\ j &= 7-1 \\ &= 6 \end{aligned}$$

Then,

$$\begin{aligned} t &= j+2 \\ &= 6+2 \\ &= 8 \end{aligned}$$

So, we see that Tom scored 8 on his test and Jane scored 6.

This problem is an example of a simple *mathematical model*. We took a problem and we able to write a set of equations that represented the problem, mathematically. The solution of the equations then gave the solution to the problem.

10.7.2 Problem Solving Strategy

The purpose of this section is to teach you the skills that you need to be able to take a problem and formulate it mathematically, in order to solve it. The general steps to follow are:

1. Read ALL of it !
2. Find out what is requested.
3. Let the requested be a variable e.g. x.
4. Rewrite the information given in terms of x. That is, translate the words into algebraic language. This is the reponse
5. Set up an equation (i.e. a mathematical sentence or model) to solve the required variable.
6. Solve the equation algebraically to find the result.

Important: Follow the three R's and solve the problem... **Request** - **Response** - **Result**

10.7.3 Application of Mathematical Modelling

Worked Example 44: Mathematical Modelling: One variable

Question: A fruit shake costs R2,00 more than a chocolate milkshake. If three fruit shakes and 5 chocolate milkshakes cost R78,00, determine the individual prices.

Answer

Step 1 : Summarise the information in a table

	Price	number	Total
Fruit	$x+2$	3	$3(x+2)$
Chocolate	x	5	$5x$

Step 2 : Set up an algebraic equation

$$3(x+2)+5x = 78$$

Step 3 : Solve the equation

$$\begin{aligned} 3x+6+5x &= 78 \\ 8x &= 72 \\ x &= 9 \end{aligned}$$

Step 4 : Present the final answer

Chocolate milkshake costs R 9,00 and the Fruitshake costs R 11,00

Worked Example 45: Mathematical Modelling: Two variables

Question: Three rulers and two pens cost R 21,00. One ruler and one pen cost R 8,00. Find the cost of one ruler and one pen

Answer

Step 1 : Translate the problem using variables

Let the cost of one ruler be x rand and the cost of one pen be y rand.

Step 2 : Rewrite the information in terms of the variables

$$3x+2y = 21 \qquad (10.4)$$

$$x+y = 8 \qquad (10.5)$$

Step 3 : Solve the equations simultaneously

First solve the second equation for y:

$$y = 8 - x$$

and substitute the result into the first equation:

$$\begin{aligned} 3x+2(8-x) &= 21 \\ 3x+16-2x &= 21 \\ x &= 5 \end{aligned}$$

therefore

$$\begin{aligned} y &= 8-5 \\ y &= 3 \end{aligned}$$

Step 4 : Present the final answers

one Ruler costs R 5,00 and one Pen costs R 3,00

?

Exercise: Mathematical Models

1. Stephen has 1 l of a mixture containing 69% of salt. How much water must Stephen add to make the mixture 50% salt? Write your answer as a fraction.
2. The diagonal of a rectangle is 25 cm more than its width. The length of the rectangle is 17 cm more than its width. What are the dimensions of the rectangle?
3. The sum of 27 and 12 is 73 more than an unknown number. Find the unknown number.
4. The two smaller angles in a right-angled triangle are in the ratio of 1:2. What are the sizes of the two angles?
5. George owns a bakery that specialises in wedding cakes. For each wedding cake, it costs George R150 for ingredients, R50 for overhead, and R5 for advertising. George's wedding cakes cost R400 each. As a percentage of George's costs, how much profit does he make for each cake sold?
6. If 4 times a number is increased by 7, the result is 15 less than the square of the number. Find the numbers that satisfy this statement, by formulating an equation and then solving it.
7. The length of a rectangle is 2 cm more than the width of the rectangle. The perimeter of the rectangle is 20 cm. Find the length and the width of the rectangle.

10.7.4 End of Chapter Exercises

1. What are the roots of the quadratic equation $x^2 - 3x + 2 = 0$?
2. What are the solutions to the equation $x^2 + x = 6$?
3. In the equation $y = 2x^2 - 5x - 18$, which is a value of x when $y = 0$?
4. Manuel has 5 more CDs than Pedro has. Bob has twice as many CDs as Manuel has. Altogether the boys have 63 CDs. Find how many CDs each person has.
5. Seven-eighths of a certain number is 5 more than one-third of the number. Find the number.
6. A man runs to a telephone and back in 15 minutes. His speed on the way to the telephone is 5 m/s and his speed on the way back is 4 m/s. Find the distance to the telephone.
7. Solve the inequality and then answer the questions:
   $\frac{x}{3} - 14 > 14 - \frac{x}{4}$
   (a) If $x \epsilon R$, write the solution in interval notation.
   (b) if $x \epsilon Z$ and $x < 51$, write the solution as a set of integers.
8. Solve for a: $\frac{1-a}{2} - \frac{2-a}{3} > 1$
9. Solve for x: $x - 1 = \frac{42}{x}$
10. Solve for x and y: $7x + 3y = 13$ and $2x - 3y = -4$

10.8 Introduction to Functions and Graphs

Functions are mathematical building blocks for designing machines, predicting natural disasters, curing diseases, understanding world economies and for keeping aeroplanes in the air. Functions can take input from many variables, but always give the same answer, unique to that function. It is the fact that you always get the same answer from a set of inputs, which is what makes functions special.

A major advantage of functions is that they allow us to *visualise* equations in terms of a *graph*. A graph is an accurate drawing of a function and is much easier to read than lists of numbers. In this chapter we will learn how to understand and create real valued functions, how to read graphs and how to draw them.

Despite their use in the problems facing humanity, functions also appear on a day-to-day level, so they are worth learning about. A function is always *dependent* on one or more variables, like time, distance or a more abstract quantity.

10.9 Functions and Graphs in the Real-World

Some typical examples of functions you may already have met include:-

- how much money you have, as a function of time. You never have more than one amount of money at any time because you can always add everything to give one number. By understanding how your money changes over time, you can plan to spend your money sensibly. Businesses find it very useful to *plot the graph* of their money over time so that they can see when they are spending too much. Such observations are not always obvious from looking at the numbers alone.
- the temperature is a very complicated function because it has so many inputs, including; the time of day, the season, the amount of clouds in the sky, the strength of the wind, where you are and many more. But the important thing is that there is only one temperature when you measure it. By understanding how the temperature is effected by these things, you can plan for the day.
- where you are is a function of time, because you cannot be in two places at once! If you were to *plot the graphs* of where two people are as a function of time, if the lines cross it means that the two people meet each other at that time. This idea is used in *logistics*, an area of mathematics that tries to plan where people and items are for businesses.
- your weight is a function of how much you eat and how much exercise you do, but everybody has a different function so that is why people are all different sizes.

10.10 Recap

The following should be familiar.

10.10.1 Variables and Constants

In section 2.4 (page 8), we were introduced to variables and constants. To recap, a *variable* can take any value in some set of numbers, so long is the equation is consistent. Most often, a variable will be written as a letter.

A *constant* has a fixed value. The number 1 is a constant. Sometimes letters are used to represent constants, as its easier to work with.

Activity :: Investigation : Variables and Constants
In the following expressions, identify the variables and the constants:

1. $2x^2 = 1$
2. $3x + 4y = 7$
3. $y = \frac{-5}{x}$
4. $y = 7^x - 2$

10.10.2 Relations and Functions

In earlier grades, you saw that variables can be *related* to each other. For example, Alan is two years older than Nathan. Therefore the relationship between the ages of Alan and Nathan can be written as $A = N + 2$, where A is Alan's age and N is Nathan's age.

In general, a *relation* is an equation which relates two variables. For example, $y = 5x$ and $y^2 + x^2 = 5$ are relations. In both examples x and y are variables and 5 is a constant, but for a given value of x the value of y will be very different in each relation.

Besides writing relations as equations, they can also be represented as words, tables and graphs. Instead of writing $y = 5x$, we could also say "y is always five times as big as x". We could also give the following table:

x	$y = 5x$
2	10
6	30
8	40
13	65
15	75

Activity :: Investigation : Relations and Functions
Complete the following table for the given functions:

x	$y = x$	$y = 2x$	$y = x + 2$
1			
2			
3			
50			
100			

10.10.3 The Cartesian Plane

When working with real valued functions, our major tool is drawing graphs. In the first place, if we have two real variables, x and y, then we can assign values to them simultaneously. That is, we can say "let x be 5 and y be 3". Just as we write "let $x = 5$" for "let x be 5", we have the shorthand notation "let $(x, y) = (5, 3)$" for "let x be 5 and y be 3". We usually think of the real numbers as an infinitely long line, and picking a number as putting a dot on that line. If we want to pick *two* numbers at the same time, we can do something similar, but now we must use two dimensions. What we do is use two lines, one for x and one for y, and rotate the one for y, as in Figure 10.1. We call this the *Cartesian plane*.

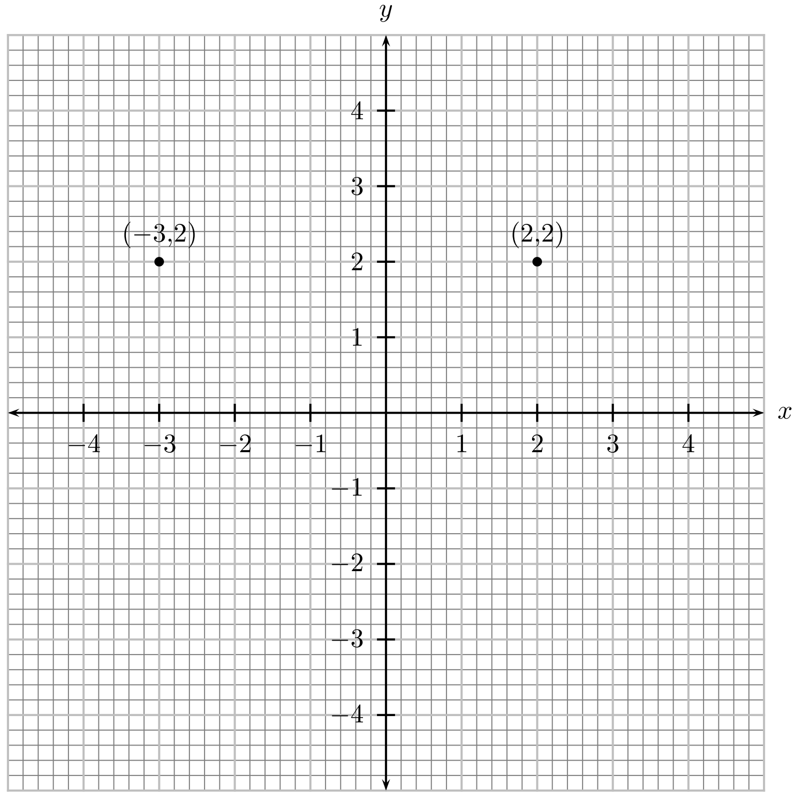


Figure 10.1: The Cartesian plane is made up of an $x-$axis (horizontal) and a $y-$axis (vertical).

10.10.4 Drawing Graphs

In order to draw the graph of a function, we need to calculate a few points. Then we plot the points on the Cartesian Plane and join the points with a smooth line.

The great beauty of doing this is that it allows us to "draw" functions, in a very abstract way. Assume that we were investigating the properties of the function $f(x) = 2x$. We could then consider all the points (x, y) such that $y = f(x)$, i.e. $y = 2x$. For example, $(1, 2), (2.5, 5)$, and $(3, 6)$ would all be such points, whereas $(3, 5)$ would not since $5 \neq 2 \times 3$. If we put a dot at each of those points, and then at every similar one for all possible values of x, we would obtain the graph shown in

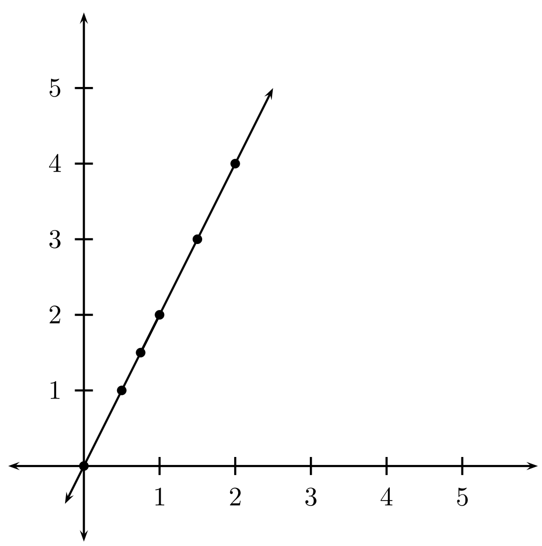


Figure 10.2: Graph of $f(x) = 2x$

The form of this graph is very pleasing – it is a simple straight line through the middle of the plane. The technique of "plotting", which we have followed here, is *the* key element in understanding functions.

Activity :: Investigation : Drawing Graphs and the Cartesian Plane
Plot the following points and draw a smooth line through them. (-6; -8),(-2; 0), (2; 8), (6; 16)

10.10.5 Notation used for Functions

Thus far you would have seen that we can use $y = 2x$ to represent a function. This notation however gets confusing when you are working with more than one function. A more general form of writing a function is to write the function as $f(x)$, where f is the function name and x is the independent variable. For example, $f(x) = 2x$ and $g(t) = 2t + 1$ are two functions.

Both notations will be used in this book.

Worked Example 46: Function notation

Question: If $f(n) = n^2 - 6n + 9$, find $f(k - 1)$ in terms of k.
Answer
Step 1 : Replace n with $k - 1$

$$\begin{aligned} f(n) &= n^2 - 6n + 9 \\ f(k-1) &= (k-1)^2 - 6(k-1) + 9 \end{aligned}$$

Step 2 : Remove brackets on RHS and simplify

$$\begin{aligned} &= k^2 - 2k + 1 - 6k + 6 + 9 \\ &= k^2 - 8k + 16 \end{aligned}$$

Worked Example 47: Function notation

Question: If $f(x) = x^2 - 4$, calculate b if $f(b) = 45$.
Answer
Step 1 : Replace x with b

$$\begin{aligned} f(b) &= b^2 - 4 \\ but f(b) &= 45 \end{aligned}$$

Step 2 : f(b) = f(b)

$$\begin{aligned} b^2 - 4 &= 45 \\ b^2 - 49 &= 0 \\ b &= +7 or - 7 \end{aligned}$$

{ExerciseRecap

1. Guess the function in the form $y = \ldots$ that has the values listed in the table.

x	1	2	3	40	50	600	700	800	900	1000
y	1	2	3	40	50	600	700	800	900	1000

2. Guess the function in the form $y = \ldots$ that has the values listed in the table.

x	1	2	3	40	50	600	700	800	900	1000
y	2	4	6	80	100	1200	1400	1600	1800	2000

3. Guess the function in the form $y = \ldots$ that has the values listed in the table.

x	1	2	3	40	50	600	700	800	900	1000
y	10	20	30	400	500	6000	7000	8000	9000	10000

4. On a Cartesian plane, plot the following points: (1,2), (2,4), (3,6), (4,8), (5,10). Join the points. Do you get a straight-line?
5. If $f(x) = x + x^2$, write out:
   (a) $f(t)$
   (b) $f(a)$
   (c) $f(1)$
   (d) $f(3)$
6. If $g(x) = x$ and $f(x) = 2x$, write out:
   (a) $f(t) + g(t)$
   (b) $f(a) - g(a)$
   (c) $f(1) + g(2)$
   (d) $f(3) + g(s)$
7. A car drives by you on a straight highway. The car is travelling 10 m every second. Complete the table below by filling in how far the car has travelled away from you after 5, 10 and 20 seconds.

Time (s)	0	1	2	5	10	20
Distance (m)	0	10	20			

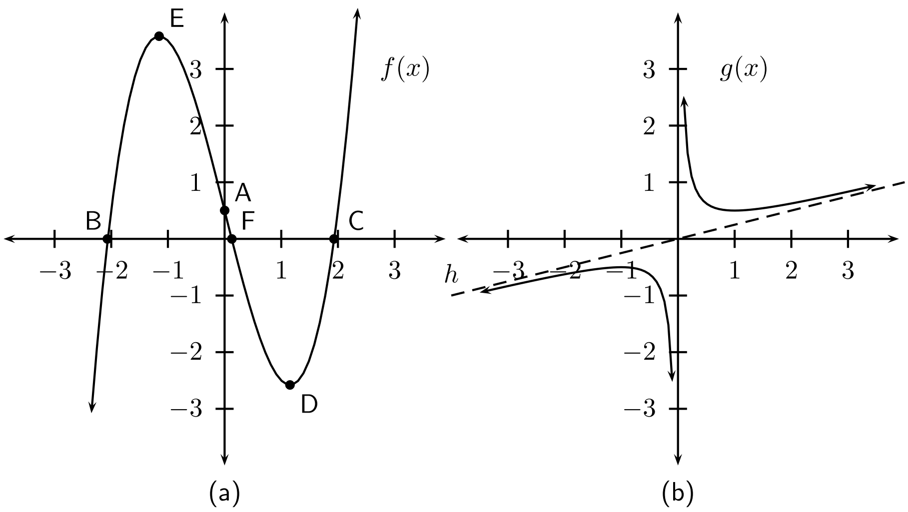
   Use the values in the table and draw a graph of distance on the y-axis and time on the x-axis.

10.11 Characteristics of Functions - All Grades

There are many characteristics of graphs that help describe the graph of any function. These properties are:

1. dependent and independent variables
2. domain and range
3. intercepts with axes
4. turning points
5. asymptotes
6. lines of symmetry
7. intervals on which the function increases/decreases
8. continuous nature of the function

Some of these words may be unfamiliar to you, but each will be clearly described. Examples of these properties are shown in Figure 10.3.



A	y-intercept
B, C, F	x-intercept
D, E	turning points

Figure 10.3: (a) Example graphs showing the characteristics of a function. (b) Example graph showing asymptotes of a function.

10.11.1 Dependent and Independent Variables

Thus far, all the graphs you have drawn have needed two values, an x-value and a y-value. The y-value is usually determined from some relation based on a given or chosen x-value. These values are given special names in mathematics. The given or chosen x-value is known as the *independent* variable, because its value can be chosen freely. The calculated y-value is known as the *dependent* variable, because its value depends on the chosen x-value.

10.11.2 Domain and Range

The *domain* of a relation is the set of all the x values for which there exists at least one y value according to that relation. The *range* is the set of all the y values, which can be obtained using at least one x value.

If the relation is of height to people, then the domain is all living people, while the range would be about 0.1 to 3 metres — no living person can have a height of 0m, and while strictly its not impossible to be taller than 3 metres, no one alive is. An important aspect of this range is that it does not contain *all* the numbers between 0.1 and 3, but only six billion of them (as many as there are people).

As another example, suppose x and y are real valued variables, and we have the relation $y = 2^x$. Then for *any* value of x, there is a value of y, so the domain of this relation is the whole set of real numbers. However, we know that no matter what value of x we choose, 2^x can never be less than or equal to 0. Hence the range of this function is all the real numbers strictly greater than zero.

These are two ways of writing the domain and range of a function, *set notation* and *interval notation*. Both notations are used in mathematics, so you should be familiar with each.

Set Notation

A set of certain x values has the following form:

$$\{x : \text{conditions, more conditions}\} \quad (10.6)$$

We read this notation as "the set of all x values where all the conditions are satisfied". For example, the set of all positive real numbers can be written as $\{x : x \in \mathbb{R}, x > 0\}$ which reads as "the set of all x values where x is a real number and is greater than zero".

Interval Notation

Here we write an interval in the form '*lower bracket, lower number, comma, upper number, upper bracket*'. We can use two types of brackets, square ones $[,]$ or round ones $(,)$. A square bracket means including the number at the end of the interval whereas a round bracket means excluding the number at the end of the interval. It is important to note that this notation can only be used for all real numbers in an interval. It cannot be used to describe integers in an interval or rational numbers in an interval.

So if x is a real number greater than 2 and less than or equal to 8, then x is any number in the interval

$$(2,8] \quad (10.7)$$

It is obvious that 2 is the lower number and 8 the upper number. The round bracket means 'excluding 2', since x is greater than 2, and the square bracket means 'including 8' as x is less than or equal to 8.

10.11.3 Intercepts with the Axes

The *intercept* is the point at which a graph intersects an axis. The x-intercepts are the points at which the graph cuts the x-axis and the y-intercepts are the points at which the graph cuts the y-axis.

In Figure 10.3(a), the A is the y-intercept and B, C and F are x-intercepts.

You **will** usually need to calculate the intercepts. The two most important things to remember is that at the x-intercept, $y = 0$ and at the y-intercept, $x = 0$.

For example, calculate the intercepts of $y = 3x + 5$. For the y-intercept, $x = 0$. Therefore the y-intercept is $y_{int} = 3(0) + 5 = 5$. For the x-intercept, $y = 0$. Therefore the x-intercept is found from $0 = 3x_{int} + 5$, giving $x_{int} = -\frac{5}{3}$.

10.11.4 Turning Points

Turning points only occur for graphs of functions that whose highest power is greater than 1. For example, graphs of the following functions will have turning points.

$$\begin{aligned} f(x) &= 2x^2 - 2 \\ g(x) &= x^3 - 2x^2 + x - 2 \\ h(x) &= \frac{2}{3}x^4 - 2 \end{aligned}$$

There are two types of turning points: a minimal turning point and a maximal turning point. A minimal turning point is a point on the graph where the graph stops decreasing in value and starts increasing in value and a maximal turning point is a point on the graph where the graph stops increasing in value and starts decreasing. These are shown in Figure 10.4.

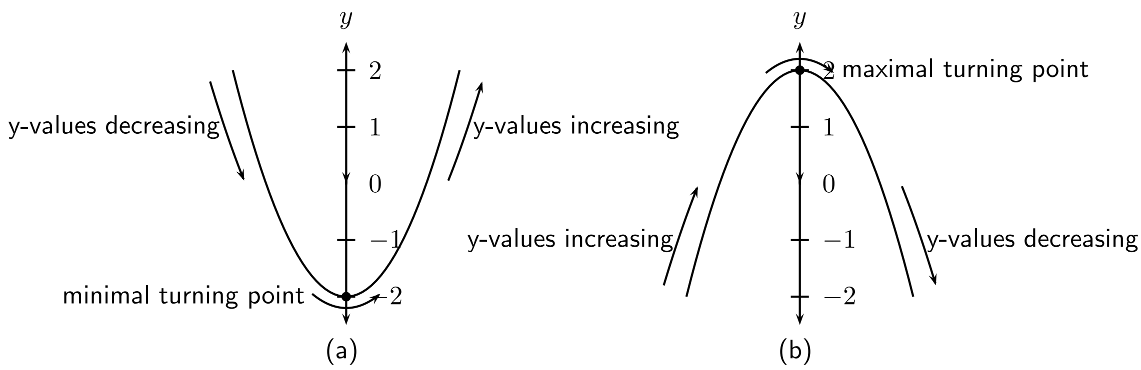


Figure 10.4: (a) Maximal turning point. (b) Minimal turning point.

In Figure 10.3(a), E is a maximal turning point and D is a minimal turning point.

10.11.5 Asymptotes

An asymptote is a straight or curved line, which the graph of a function will approach, but never touch.

In Figure 10.3(b), the y-axis and line h are both asymptotes as the graph approaches both these lines, but never touches them.

10.11.6 Lines of Symmetry

Graphs look the same on either side of lines of symmetry. These lines include the x- and y-axes. For example, in Figure 10.5 is symmetric about the y-axis. This is described as the axis of symmetry.

10.11.7 Intervals on which the Function Increases/Decreases

In the discussion of turning points, we saw that the graph of a function can start or stop increasing or decreasing at a turning point. If the graph in Figure 10.3(a) is examined, we find that the values of the graph increase and decrease over different intervals. We see that the graph increases (i.e. that the y-values increase) from -∞ to point E, then it decreases (i.e. the y-values decrease) from point E to point D and then it increases from point D to $+\infty$.

10.11.8 Discrete or Continuous Nature of the Graph

A graph is said to be continuous if there are no breaks in the graph. For example, the graph in Figure 10.3(a) can be described as a continuous graph, while the graph in Figure 10.3(b) has a

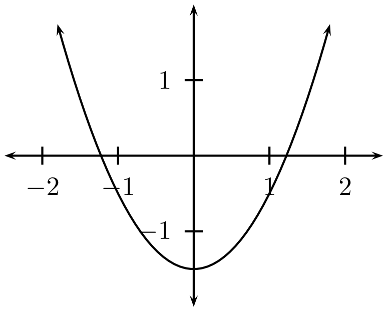


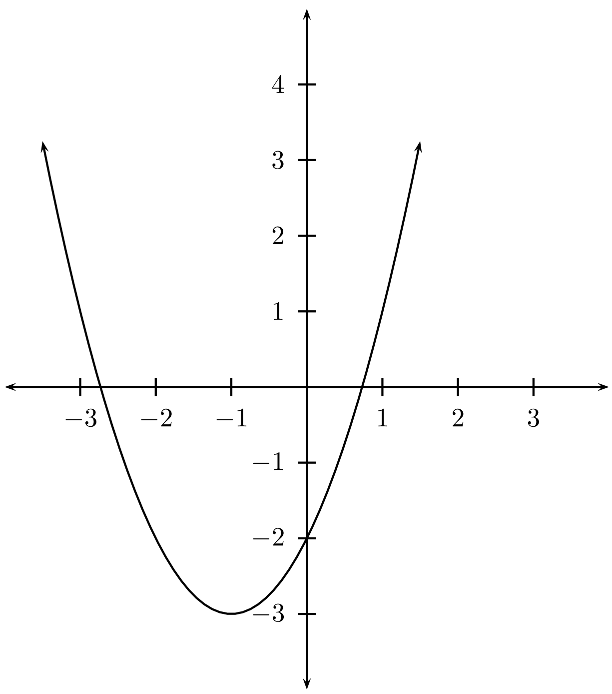
Figure 10.5: Demonstration of axis of symmetry. The y-axis is an axis of symmetry, because the graph looks the same on both sides of the y-axis.

break around the asymptotes. In Figure 10.3(b), it is clear that the graph does have a break in it around the asymptote.

?

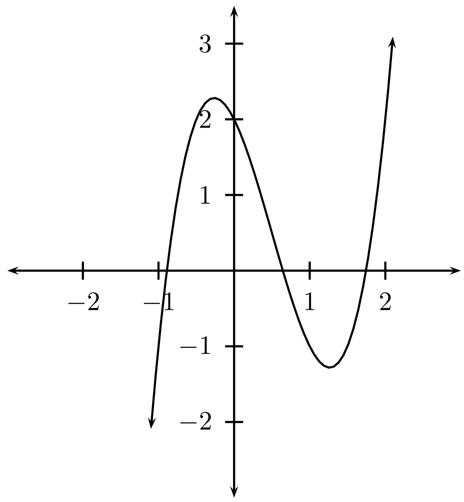
Exercise: Domain and Range

1. The domain of the function $f(x) = 2x+5$ is -3; -3; -3; 0. Determine the range of f.
2. If $g(x) = -x^2 + 5$ and x is between - 3 and 3, determine:
   (a) the domain of $g(x)$
   (b) the range of $g(x)$
3. Label, on the following graph:
   (a) the x-intercept(s)
   (b) the y-intercept(s)
   (c) regions where the graph is increasing
   (d) regions where the graph is decreasing



4. Label, on the following graph:
   (a) the x-intercept(s)
   (b) the y-intercept(s)

(c) regions where the graph is increasing
(d) regions where the graph is decreasing



10.12 Graphs of Functions

10.12.1 Functions of the form $y = ax + q$

Functions with a general form of $y = ax + q$ are called *straight line* functions. In the equation, $y = ax + q$, a and q are constants and have different effects on the graph of the function. The general shape of the graph of functions of this form is shown in Figure 10.6 for the function $f(x) = 2x + 3$.

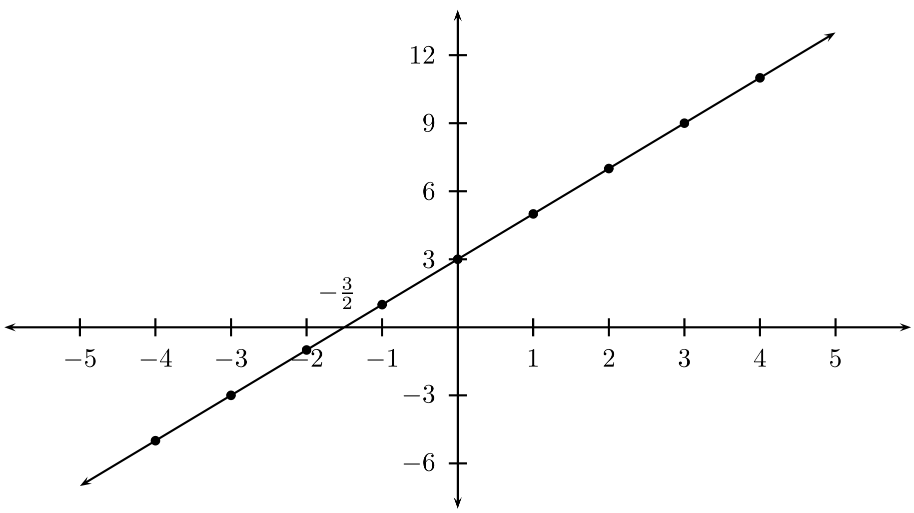


Figure 10.6: Graph of $f(x) = 2x + 3$

Activity :: Investigation : Functions of the Form $y = ax + q$

1. On the same set of axes, plot the following graphs:
   (a) $a(x) = x - 2$

(b) $b(x) = x - 1$
(c) $c(x) = x$
(d) $d(x) = x + 1$
(e) $e(x) = x + 2$

Use your results to deduce the effect of q.

2. On the same set of axes, plot the following graphs:
   (a) $f(x) = -2 \cdot x$
   (b) $g(x) = -1 \cdot x$
   (c) $h(x) = 0 \cdot x$
   (d) $j(x) = 1 \cdot x$
   (e) $k(x) = 2 \cdot x$

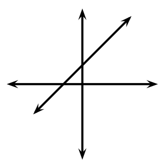
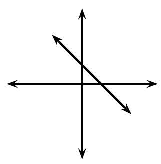
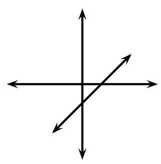
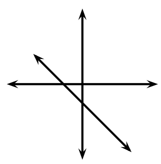
   Use your results to deduce the effect of a.

You should have found that the value of a affects the slope of the graph. As a increases, the slope of the graph increases. If $a > 0$ then the graph increases from left to right (slopes upwards). If $a < 0$ then the graph increases from right to left (slopes downwards). For this reason, a is referred to as the *slope* or *gradient* of a straight-line function.

You should have also found that the value of q affects where the graph passes through the y-axis. For this reason, q is known as the *y-intercept*.

These different properties are summarised in Table 10.1.

Table 10.1: Table summarising general shapes and positions of graphs of functions of the form $y = ax + q$.

	$a > 0$	$a < 0$
$q > 0$		
$q < 0$		

Domain and Range

For $f(x) = ax + q$, the domain is $\{x : x \in \mathbb{R}\}$ because there is no value of $x \in \mathbb{R}$ for which $f(x)$ is undefined.

The range of $f(x) = ax + q$ is also $\{f(x) : f(x) \in \mathbb{R}\}$ because there is no value of $f(x) \in \mathbb{R}$ for which $f(x)$ is undefined.

For example, the domain of $g(x) = x - 1$ is $\{x : x \in \mathbb{R}\}$ because there is no value of $x \in \mathbb{R}$ for which $g(x)$ is undefined. The range of $g(x)$ is $\{g(x) : g(x) \in \mathbb{R}\}$.

Intercepts

For functions of the form, $y = ax + q$, the details of calculating the intercepts with the x and y axis is given.

The y-intercept is calculated as follows:

$$\begin{aligned} y &= ax + q && (10.8)\\ y_{int} &= a(0) + q && (10.9)\\ &= q && (10.10) \end{aligned}$$

For example, the y-intercept of $g(x) = x - 1$ is given by setting $x = 0$ to get:

$$\begin{aligned} g(x) &= x - 1\\ y_{int} &= 0 - 1\\ &= -1 \end{aligned}$$

The x-intercepts are calculated as follows:

$$\begin{aligned} y &= ax + q && (10.11)\\ 0 &= a \cdot x_{int} + q && (10.12)\\ a \cdot x_{int} &= -q && (10.13)\\ x_{int} &= -\frac{q}{a} && (10.14) \end{aligned}$$

For example, the x-intercepts of $g(x) = x - 1$ is given by setting $y = 0$ to get:

$$\begin{aligned} g(x) &= x - 1\\ 0 &= x_{int} - 1\\ x_{int} &= 1 \end{aligned}$$

Turning Points

The graphs of straight line functions do not have any turning points.

Axes of Symmetry

The graphs of straight-line functions do not, generally, have any axes of symmetry.

Sketching Graphs of the Form $f(x) = ax + q$

In order to sketch graphs of the form, $f(x) = ax + q$, we need to determine three characteristics:

1. sign of a
2. y-intercept
3. x-intercept

Only two points are needed to plot a straight line graph. The easiest points to use are the x-*intercept* (where the line cuts the x-axis) and the y-intercept.

For example, sketch the graph of $g(x) = x - 1$. Mark the intercepts.

Firstly, we determine that $a > 0$. This means that the graph will have an upward slope.

The y-intercept is obtained by setting $x = 0$ and was calculated earlier to be $y_{int} = -1$. The x-intercept is obtained by setting $y = 0$ and was calculated earlier to be $x_{int} = 1$.

Worked Example 48: Drawing a straight line graph

Question: Draw the graph of $y = 2x + 2$

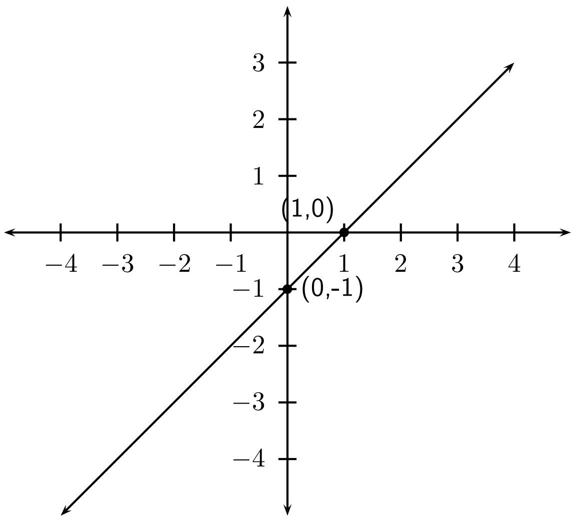


Figure 10.7: Graph of the function $g(x) = x - 1$

Answer

Step 1 : Find the y-intercept

For the intercept on the y-axis, let $x = 0$

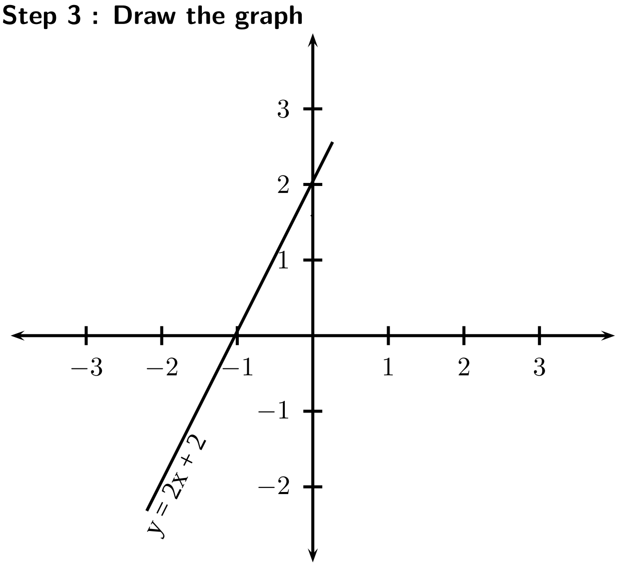
$$\begin{aligned} y &= 2(0) + 2 \\ &= 2 \end{aligned}$$

Step 2 : Find the x-intercept

For the intercept on the x-axis, let $y = 0$

$$\begin{aligned} 0 &= 2x + 2 \\ 2x &= -2 \\ x &= -1 \end{aligned}$$

Step 3 : Draw the graph



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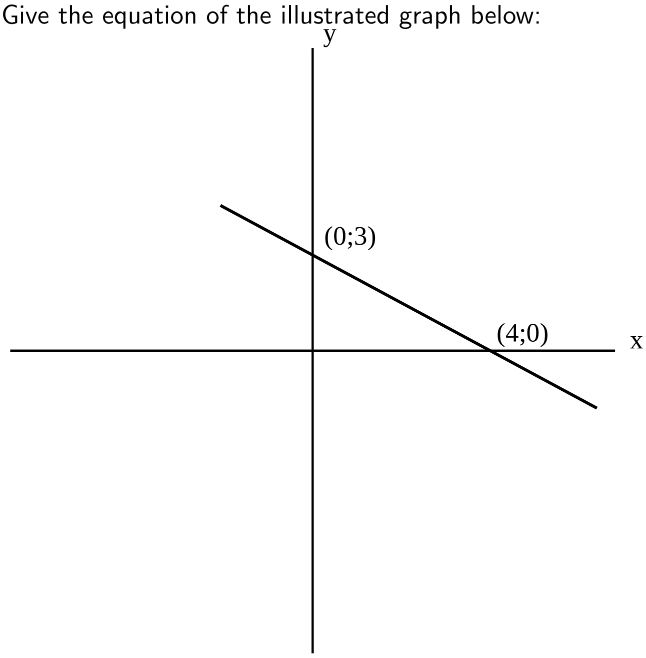
Exercise: Intercepts

1. List the y-intercepts for the following straight-line graphs:
   (a) $y = x$
   (b) $y = x - 1$
   (c) $y = 2x - 1$
   (d) $y + 1 = 2x$
2. Give the equation of the illustrated graph below:



3. Sketch the following relations on the same set of axes, clearly indicating the intercepts with the axes as well as the co-ordinates of the point of interception on the graph: $x + 2y - 5 = 0$ and $3x - y - 1 = 0$

10.12.2 Functions of the Form $y = ax^2 + q$

The general shape and position of the graph of the function of the form $f(x) = ax^2 + q$ is shown in Figure 10.8.

Activity :: Investigation : Functions of the Form $y = ax^2 + q$

1. On the same set of axes, plot the following graphs:
   (a) $a(x) = -2 \cdot x^2 + 1$
   (b) $b(x) = -1 \cdot x^2 + 1$
   (c) $c(x) = 0 \cdot x^2 + 1$
   (d) $d(x) = 1 \cdot x^2 + 1$
   (e) $e(x) = 2 \cdot x^2 + 1$

   Use your results to deduce the effect of a.
2. On the same set of axes, plot the following graphs:
   (a) $f(x) = x^2 - 2$
   (b) $g(x) = x^2 - 1$

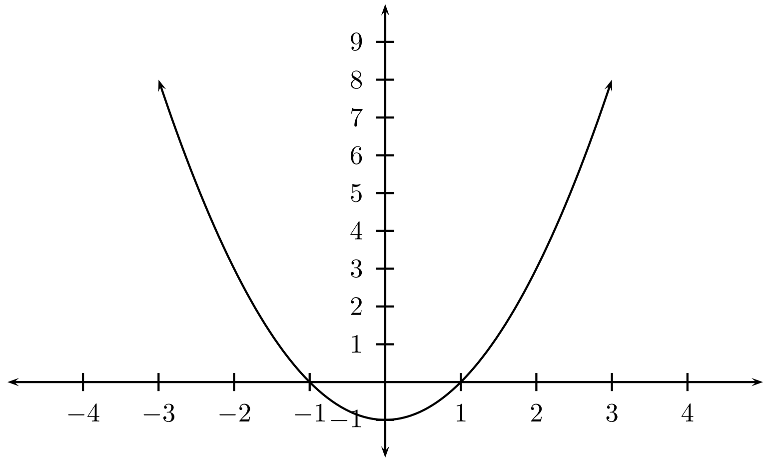


Figure 10.8: Graph of the $f(x) = x^2 - 1$.

(c) $h(x) = x^2 + 0$

(d) $j(x) = x^2 + 1$

(e) $k(x) = x^2 + 2$

Use your results to deduce the effect of q.

Complete the following table of values for the functions a to k to help with drawing the required graphs in this activity:

x	-2	-1	0	1	2
$a(x)$					
$b(x)$					
$c(x)$					
$d(x)$					
$e(x)$					
$f(x)$					
$g(x)$					
$h(x)$					
$j(x)$					
$k(x)$					

From your graphs, you should have found that a affects whether the graph makes a smile or a frown. If $a < 0$, the graph makes a frown and if $a > 0$ then the graph makes a smile. This is shown in Figure 10.9.

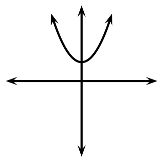
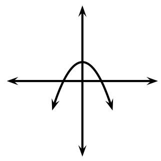
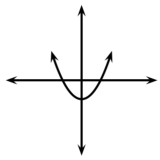
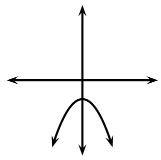
$a > 0$ (a positive smile) $a < 0$ (a negative frown)

Figure 10.9: Distinctive shape of graphs of a parabola if $a > 0$ and $a < 0$.

You should have also found that the value of q affects whether the turning point is to the left of the y-axis ($q > 0$) or to the right of the y-axis ($q < 0$).

These different properties are summarised in Table **??**.

Table 10.2: Table summarising general shapes and positions of functions of the form $y = ax^2 + q$.

	$a > 0$	$a < 0$
$q > 0$		
$q < 0$		

Domain and Range

For $f(x) = ax^2 + q$, the domain is $\{x : x \in \mathbb{R}\}$ because there is no value of $x \in \mathbb{R}$ for which $f(x)$ is undefined.

The range of $f(x) = ax^2 + q$ depends on whether the value for a is positive or negative. We will consider these two cases separately.

If $a > 0$ then we have:

$$\begin{aligned} x^2 &\geq 0 \quad \text{(The square of an expression is always positive)} \\ ax^2 &\geq 0 \quad \text{(Multiplication by a positive number maintains the nature of the inequality)} \\ ax^2 + q &\geq q \\ f(x) &\geq q \end{aligned}$$

This tells us that for all values of x, $f(x)$ is always greater than q. Therefore if $a > 0$, the range of $f(x) = ax^2 + q$ is $\{f(x) : f(x) \in [q,\infty)\}$.

Similarly, it can be shown that if $a < 0$ that the range of $f(x) = ax^2 + q$ is $\{f(x) : f(x) \in (-\infty,q]\}$. This is left as an exercise.

For example, the domain of $g(x) = x^2 + 2$ is $\{x : x \in \mathbb{R}\}$ because there is no value of $x \in \mathbb{R}$ for which $g(x)$ is undefined. The range of $g(x)$ can be calculated as follows:

$$\begin{aligned} x^2 &\geq 0 \\ x^2 + 2 &\geq 2 \\ g(x) &\geq 2 \end{aligned}$$

Therefore the range is $\{g(x) : g(x) \in [2,\infty)\}$.

Intercepts

For functions of the form, $y = ax^2 + q$, the details of calculating the intercepts with the x and y axis is given.

The y-intercept is calculated as follows:

$$y = ax^2 + q \quad (10.15)$$
$$y_{int} = a(0)^2 + q \quad (10.16)$$
$$= q \quad (10.17)$$

For example, the y-intercept of $g(x) = x^2 + 2$ is given by setting $x = 0$ to get:

$$\begin{aligned} g(x) &= x^2 + 2 \\ y_{int} &= 0^2 + 2 \\ &= 2 \end{aligned}$$

The x-intercepts are calculated as follows:

$$y = ax^2 + q \quad (10.18)$$
$$0 = ax_{int}^2 + q \quad (10.19)$$
$$ax_{int}^2 = -q \quad (10.20)$$
$$x_{int} = \pm\sqrt{-\frac{q}{a}} \quad (10.21)$$

However, (10.21) is only valid if $-\frac{q}{a} > 0$ which means that either $q < 0$ or $a < 0$. This is consistent with what we expect, since if $q > 0$ and $a > 0$ then $-\frac{q}{a}$ is negative and in this case the graph lies above the x-axis and therefore does not intersect the x-axis. If however, $q > 0$ and $a < 0$, then $-\frac{q}{a}$ is positive and the graph is hat shaped and should have two x-intercepts. Similarly, if $q < 0$ and $a > 0$ then $-\frac{q}{a}$ is also positive, and the graph should intersect with the x-axis.

For example, the x-intercepts of $g(x) = x^2 + 2$ is given by setting $y = 0$ to get:

$$\begin{aligned} g(x) &= x^2 + 2 \\ 0 &= x_{int}^2 + 2 \\ -2 &= x_{int}^2 \end{aligned}$$

which is not real. Therefore, the graph of $g(x) = x^2 + 2$ does not have any x-intercepts.

Turning Points

The turning point of the function of the form $f(x) = ax^2 + q$ is given by examining the range of the function. We know that if $a > 0$ then the range of $f(x) = ax^2 + q$ is $\{f(x) : f(x) \in [q,\infty)\}$ and if $a < 0$ then the range of $f(x) = ax^2 + q$ is $\{f(x) : f(x) \in (-\infty,q]\}$.

So, if $a > 0$, then the lowest value that $f(x)$ can take on is q. Solving for the value of x at which $f(x) = q$ gives:

$$\begin{aligned} q &= ax_{tp}^2 + q \\ 0 &= ax_{tp}^2 \\ 0 &= x_{tp}^2 \\ x_{tp} &= 0 \end{aligned}$$

$\therefore x = 0$ at $f(x) = q$. The co-ordinates of the (minimal) turning point is therefore $(0; q)$.

Similarly, if $a < 0$, then the highest value that $f(x)$ can take on is q and the co-ordinates of the (maximal) turning point is $(0; q)$.

Axes of Symmetry

There is one axis of symmetry for the function of the form $f(x) = ax^2 + q$ that passes through the turning point. Since the turning point lies on the y-axis, the axis of symmetry is the y-axis.

Sketching Graphs of the Form $f(x) = ax^2 + q$

In order to sketch graphs of the form, $f(x) = ax^2 + q$, we need to calculate determine four characteristics:

1. sign of a

2. domain and range
3. turning point
4. y-intercept
5. x-intercept

For example, sketch the graph of $g(x) = -\frac{1}{2}x^2 - 3$. Mark the intercepts, turning point and axis of symmetry.

Firstly, we determine that $a < 0$. This means that the graph will have a maximal turning point.

The domain of the graph is $\{x : x \in \mathbb{R}\}$ because $f(x)$ is defined for all $x \in \mathbb{R}$. The range of the graph is determined as follows:

$$\begin{aligned} x^2 &\geq 0 \\ -\frac{1}{2}x^2 &\leq 0 \\ -\frac{1}{2}x^2 - 3 &\leq -3 \\ \therefore f(x) &\leq -3 \end{aligned}$$

Therefore the range of the graph is $\{f(x) : f(x) \in (-\infty, -3]\}$.

Using the fact that the maximum value that $f(x)$ achieves is -3, then the y-coordinate of the turning point is -3. The x-coordinate is determined as follows:

$$\begin{aligned} -\frac{1}{2}x^2 - 3 &= -3 \\ -\frac{1}{2}x^2 - 3 + 3 &= 0 \\ -\frac{1}{2}x^2 &= 0 \\ \text{Divide both sides by } -\tfrac{1}{2}\text{:} \quad x^2 &= 0 \\ \text{Take square root of both sides:} \quad x &= 0 \\ \therefore \quad x &= 0 \end{aligned}$$

The coordinates of the turning point are: $(0, -3)$.

The y-intercept is obtained by setting $x = 0$. This gives:

$$\begin{aligned} y_{int} &= -\frac{1}{2}(0)^2 - 3 \\ &= -\frac{1}{2}(0) - 3 \\ &= -3 \end{aligned}$$

The x-intercept is obtained by setting $y = 0$. This gives:

$$\begin{aligned} 0 &= -\frac{1}{2}x_{int}^2 - 3 \\ 3 &= -\frac{1}{2}x_{int}^2 \\ -3 \cdot 2 &= x_{int}^2 \\ -6 &= x_{int}^2 \end{aligned}$$

which is not real. Therefore, there are no x-intercepts.

We also know that the axis of symmetry is the y-axis.

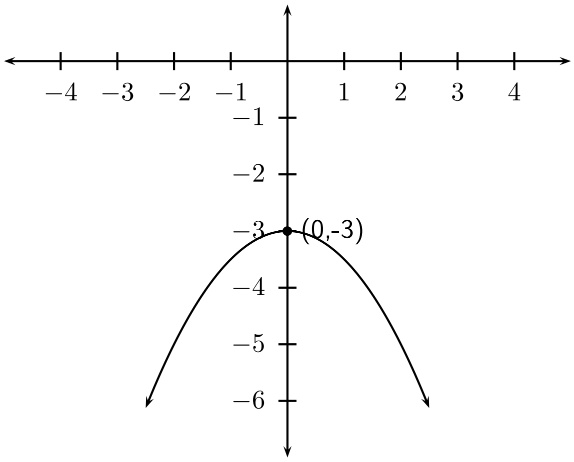
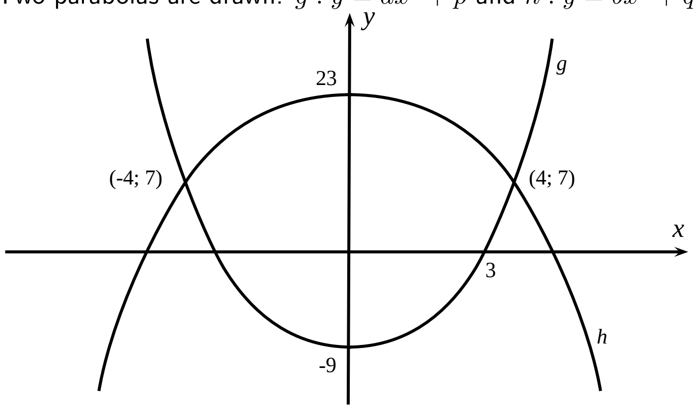


Figure 10.10: Graph of the function $f(x) = -\frac{1}{2}x^2 - 3$

?

Exercise: Parabolas

1. Show that if $a < 0$ that the range of $f(x) = ax^2+q$ is $\{f(x) : f(x) \in (-\infty,q]\}$.
2. Draw the graph of the function $y = -x^2 + 4$ showing all intercepts with the axes.
3. Two parabolas are drawn: $g : y = ax^2 + p$ and $h : y = bx^2 + q$.



   (a) Find the values of a and p.
   (b) Find the values of b and q.
   (c) Find the values of x for which $ax^2 + p \geq bx^2 + q$.
   (d) For what values of x is g increasing ?

10.12.3 Functions of the Form $y = \frac{a}{x} + q$

Functions of the form $y = \frac{a}{x} + q$ are known as *hyperbolic* functions. The general form of the graph of this function is shown in Figure 10.11.

Activity :: Investigation : Functions of the Form $y = \frac{a}{x} + q$

1. On the same set of axes, plot the following graphs:
   (a) $a(x) = \frac{-2}{x} + 1$
   (b) $b(x) = \frac{-1}{x} + 1$

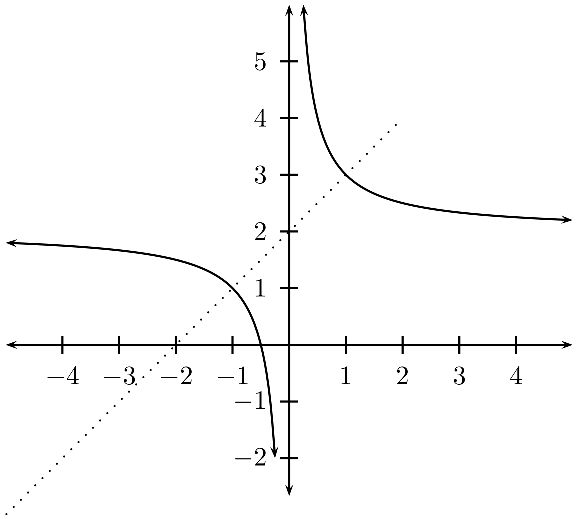


Figure 10.11: General shape and position of the graph of a function of the form $f(x) = \frac{a}{x} + q$.

(c) $c(x) = \frac{0}{x} + 1$
(d) $d(x) = \frac{+1}{x} + 1$
(e) $e(x) = \frac{+2}{x} + 1$

Use your results to deduce the effect of a.

2. On the same set of axes, plot the following graphs:
   (a) $f(x) = \frac{1}{x} - 2$
   (b) $g(x) = \frac{1}{x} - 1$
   (c) $h(x) = \frac{1}{x} + 0$
   (d) $j(x) = \frac{1}{x} + 1$
   (e) $k(x) = \frac{1}{x} + 2$

   Use your results to deduce the effect of q.

You should have found that the value of a affects whether the graph is located in the first and third quadrants of Cartesian plane.

You should have also found that the value of q affects whether the graph lies above the x-axis ($q > 0$) or below the x-axis ($q < 0$).

These different properties are summarised in Table 10.3. The axes of symmetry for each graph are shown as a dashed line.

Domain and Range

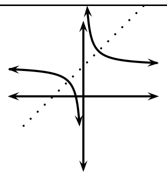
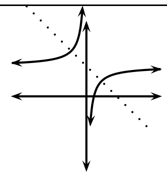
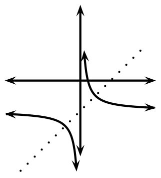
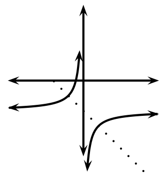
For $y = \frac{a}{x} + q$, the function is undefined for $x = 0$. The domain is therefore $\{x : x \in \mathbb{R}, x \neq 0\}$.

We see that $y = \frac{a}{x} + q$ can be re-written as:

$$\begin{aligned} y &= \frac{a}{x} + q \\ y - q &= \frac{a}{x} \\ \text{If } x \neq 0 \text{ then: } \quad (y - q)(x) &= a \\ x &= \frac{a}{y - q} \end{aligned}$$

This shows that the function is undefined at $y = q$. Therefore the range of $f(x) = \frac{a}{x} + q$ is $\{f(x) : f(x) \in (-\infty, q) \cup (q, \infty)\}$.

Table 10.3: Table summarising general shapes and positions of functions of the form $y = \frac{a}{x} + q$. The axes of symmetry are shown as dashed lines.

	$a > 0$	$a < 0$
$q > 0$		
$q < 0$		

For example, the domain of $g(x) = \frac{2}{x} + 2$ is $\{x : x \in \mathbb{R}, x \neq 0\}$ because $g(x)$ is undefined at $x = 0$.

$$\begin{aligned} y &= \frac{2}{x} + 2 \\ (y-2) &= \frac{2}{x} \\ \text{If } x \neq 0 \text{ then: } \quad x(y-2) &= 2 \\ x &= \frac{2}{y-2} \end{aligned}$$

We see that $g(x)$ is undefined at $y = 2$. Therefore the range is $\{g(x) : g(x) \in (-\infty,2) \cup (2,\infty)\}$.

Intercepts

For functions of the form, $y = \frac{a}{x} + q$, the intercepts with the x and y axis is calculated by setting $x = 0$ for the y-intercept and by setting $y = 0$ for the x-intercept.

The y-intercept is calculated as follows:

$$y = \frac{a}{x} + q \tag{10.22}$$

$$y_{int} = \frac{a}{0} + q \tag{10.23}$$

which is undefined. Therefore there is no y-intercept.

For example, the y-intercept of $g(x) = \frac{2}{x} + 2$ is given by setting $x = 0$ to get:

$$\begin{aligned} y &= \frac{2}{x} + 2 \\ y_{int} &= \frac{2}{0} + 2 \end{aligned}$$

which is undefined.

The x-intercepts are calculated by setting $y = 0$ as follows:

$$\begin{aligned} y &= \frac{a}{x} + q & (10.24)\\ 0 &= \frac{a}{x_{int}} + q & (10.25)\\ \frac{a}{x_{int}} &= -q & (10.26)\\ a &= -q(x_{int}) & (10.27)\\ x_{int} &= \frac{a}{-q} & (10.28)\\ & & (10.29) \end{aligned}$$

For example, the x-intercept of $g(x) = \frac{2}{x} + 2$ is given by setting $x = 0$ to get:

$$\begin{aligned} y &= \frac{2}{x} + 2\\ 0 &= \frac{2}{x_{int}} + 2\\ -2 &= \frac{2}{x_{int}}\\ -2(x_{int}) &= 2\\ x_{int} &= \frac{2}{-2}\\ x_{int} &= -1 \end{aligned}$$

Asymptotes

There are two asymptotes for functions of the form $y = \frac{a}{x} + q$. They are determined by examining the domain and range.

We saw that the function was undefined at $x = 0$ and for $y = q$. Therefore the asymptotes are $x = 0$ and $y = q$.

For example, the domain of $g(x) = \frac{2}{x} + 2$ is $\{x : x \in \mathbb{R}, x \neq 0\}$ because $g(x)$ is undefined at $x = 0$. We also see that $g(x)$ is undefined at $y = 2$. Therefore the range is $\{g(x) : g(x) \in (-\infty,2) \cup (2,\infty)\}$.

From this we deduce that the asymptotes are at $x = 0$ and $y = 2$.

Sketching Graphs of the Form $f(x) = \frac{a}{x} + q$

In order to sketch graphs of functions of the form, $f(x) = \frac{a}{x} + q$, we need to calculate determine four characteristics:

1. domain and range
2. asymptotes
3. y-intercept
4. x-intercept

For example, sketch the graph of $g(x) = \frac{2}{x} + 2$. Mark the intercepts and asymptotes.

We have determined the domain to be $\{x : x \in \mathbb{R}, x \neq 0\}$ and the range to be $\{g(x) : g(x) \in (-\infty,2) \cup (2,\infty)\}$. Therefore the asymptotes are at $x = 0$ and $y = 2$.

There is no y-intercept and the x-intercept is $x_{int} = -1$.

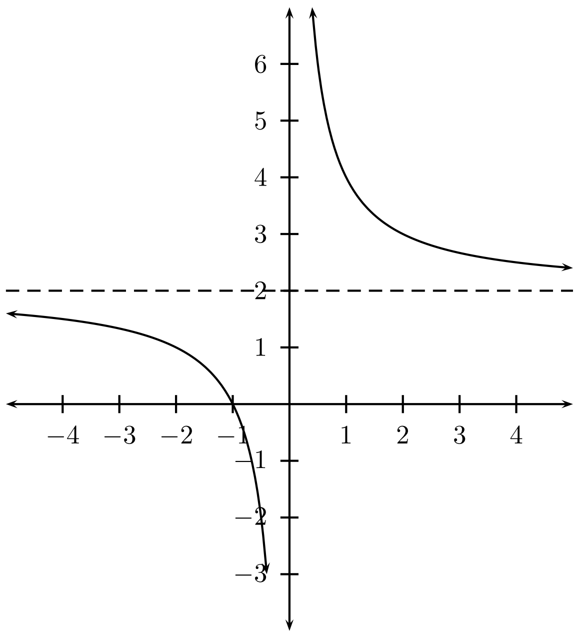


Figure 10.12: Graph of $g(x) = \frac{2}{x} + 2$.

Exercise: Graphs

1. Using grid paper, draw the graph of $xy = -6$.
   (a) Does the point (-2; 3) lie on the graph ? Give a reason for your answer.
   (b) Why is the point (-2; -3) not on the graph ?
   (c) If the x-value of a point on the drawn graph is $0{,}25$, what is the corresponding y-value ?
   (d) What happens to the y-values as the x-values become very large ?
   (e) With the line $y = -x$ as line of symmetry, what is the point symmetrical to (-2; 3) ?
2. Draw the graph of $xy = 8$.
   (a) How would the graph $y = \frac{8}{3} + 3$ compare with that of $xy = 8$? Explain your answer fully.
   (b) Draw the graph of $y = \frac{8}{3} + 3$ on the same set of axes.

10.12.4 Functions of the Form $y = ab^{(x)} + q$

Functions of the form $y = ab^{(x)} + q$ are known as *exponential* functions. The general shape of a graph of a function of this form is shown in Figure 10.13.

Activity :: Investigation : Functions of the Form $y = ab^{(x)} + q$

1. On the same set of axes, plot the following graphs:
   (a) $a(x) = -2 \cdot b^{(x)} + 1$
   (b) $b(x) = -1 \cdot b^{(x)} + 1$
   (c) $c(x) = -0 \cdot b^{(x)} + 1$

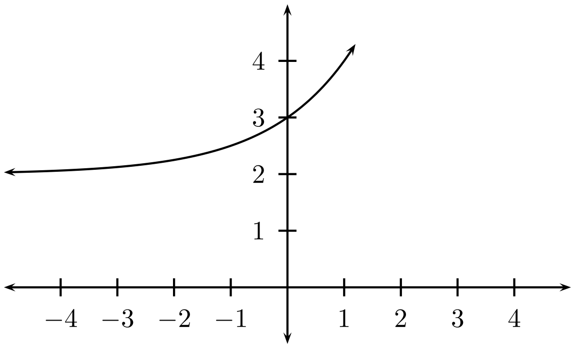


Figure 10.13: General shape and position of the graph of a function of the form $f(x) = ab^{(x)} + q$.

   (d) $d(x) = -1 \cdot b^{(x)} + 1$
   (e) $e(x) = -2 \cdot b^{(x)} + 1$

   Use your results to deduce the effect of a.

2. On the same set of axes, plot the following graphs:
   (a) $f(x) = 1 \cdot b^{(x)} - 2$
   (b) $g(x) = 1 \cdot b^{(x)} - 1$
   (c) $h(x) = 1 \cdot b^{(x)} + 0$
   (d) $j(x) = 1 \cdot b^{(x)} + 1$
   (e) $k(x) = 1 \cdot b^{(x)} + 2$

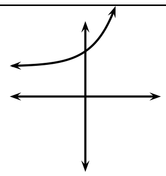
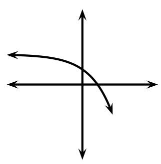
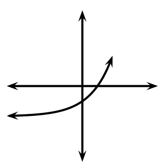
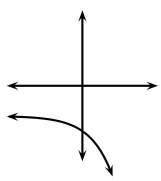
   Use your results to deduce the effect of q.

You should have found that the value of a affects whether the graph curves upwards ($a > 0$) or curves downwards ($a < 0$).

You should have also found that the value of q affects the position of the y-intercept.

These different properties are summarised in Table 10.4.

Table 10.4: Table summarising general shapes and positions of functions of the form $y = ab^{(x)} + q$.

	$a > 0$	$a < 0$
$q > 0$		
$q < 0$		

Domain and Range

For $y = ab^{(x)} + q$, the function is defined for all real values of x. Therefore, the domain is $\{x : x \in \mathbb{R}\}$.

The range of $y = ab^{(x)} + q$ is dependent on the sign of a.

If $a > 0$ then:

$$\begin{aligned} b^{(x)} &\geq 0 \\ a \cdot b^{(x)} &\geq 0 \\ a \cdot b^{(x)} + q &\geq q \\ f(x) &\geq q \end{aligned}$$

Therefore, if $a > 0$, then the range is $\{f(x) : f(x) \in [q,\infty)\}$.

If $a < 0$ then:

$$\begin{aligned} b^{(x)} &\leq 0 \\ a \cdot b^{(x)} &\leq 0 \\ a \cdot b^{(x)} + q &\leq q \\ f(x) &\leq q \end{aligned}$$

Therefore, if $a < 0$, then the range is $\{f(x) : f(x) \in (-\infty,q]\}$.

For example, the domain of $g(x) = 3 \cdot 2^x + 2$ is $\{x : x \in \mathbb{R}\}$. For the range,

$$\begin{aligned} 2^x &\geq 0 \\ 3 \cdot 2^x &\geq 0 \\ 3 \cdot 2^x + 2 &\geq 2 \end{aligned}$$

Therefore the range is $\{g(x) : g(x) \in [2,\infty)\}$.

Intercepts

For functions of the form, $y = ab^{(x)} + q$, the intercepts with the x and y axis is calculated by setting $x = 0$ for the y-intercept and by setting $y = 0$ for the x-intercept.

The y-intercept is calculated as follows:

$$\begin{aligned} y &= ab^{(x)} + q && (10.30) \\ y_{int} &= ab^{(0)} + q && (10.31) \\ &= a(1) + q && (10.32) \\ &= a + q && (10.33) \end{aligned}$$

For example, the y-intercept of $g(x) = 3 \cdot 2^x + 2$ is given by setting $x = 0$ to get:

$$\begin{aligned} y &= 3 \cdot 2^x + 2 \\ y_{int} &= 3 \cdot 2^0 + 2 \\ &= 3 + 2 \\ &= 5 \end{aligned}$$

The x-intercepts are calculated by setting $y = 0$ as follows:

$$\begin{aligned} y &= ab^{(x)} + q && (10.34) \\ 0 &= ab^{(x_{int})} + q && (10.35) \\ ab^{(x_{int})} &= -q && (10.36) \\ b^{(x_{int})} &= -\frac{q}{a} && (10.37) \end{aligned}$$

Which only has a real solution if either $a < 0$ or $q < 0$. Otherwise, the graph of the function of form $y = ab^{(x)} + q$ does not have any x-intercepts.

For example, the x-intercept of $g(x) = 3 \cdot 2^x + 2$ is given by setting $y = 0$ to get:

$$\begin{aligned} y &= 3 \cdot 2^x + 2 \\ 0 &= 3 \cdot 2^{x_{int}} + 2 \\ -2 &= 3 \cdot 2^{x_{int}} \\ 2^{x_{int}} &= \frac{-2}{3} \end{aligned}$$

which has no real solution. Therefore, the graph of $g(x) = 3 \cdot 2^x + 2$ does not have any x-intercepts.

Asymptotes

There are two asymptotes for functions of the form $y = ab^{(x)} + q$. They are determined by examining the domain and range.

We saw that the function was undefined at $x = 0$ and for $y = q$. Therefore the asymptotes are $x = 0$ and $y = q$.

For example, the domain of $g(x) = 3 \cdot 2^x + 2$ is $\{x : x \in \mathbb{R}, x \neq 0\}$ because $g(x)$ is undefined at $x = 0$. We also see that $g(x)$ is undefined at $y = 2$. Therefore the range is $\{g(x) : g(x) \in (-\infty,2) \cup (2,\infty)\}$.

From this we deduce that the asymptotes are at $x = 0$ and $y = 2$.

Sketching Graphs of the Form $f(x) = ab^{(x)} + q$

In order to sketch graphs of functions of the form, $f(x) = ab^{(x)} + q$, we need to calculate determine four characteristics:

1. domain and range
2. y-intercept
3. x-intercept

For example, sketch the graph of $g(x) = 3 \cdot 2^x + 2$. Mark the intercepts.

We have determined the domain to be $\{x : x \in \mathbb{R}\}$ and the range to be $\{g(x) : g(x) \in [2,\infty)\}$.

The y-intercept is $y_{int} = 5$ and there are no x-intercepts.

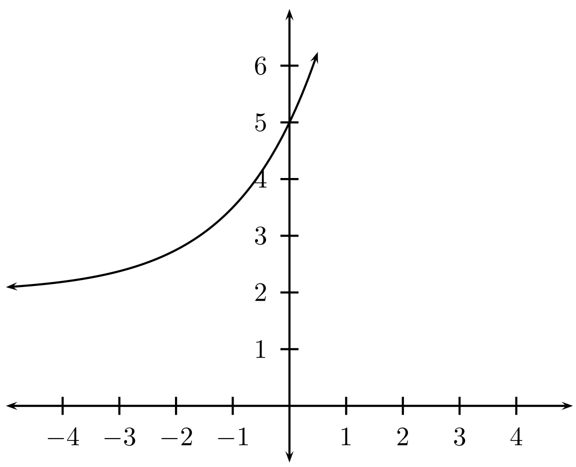
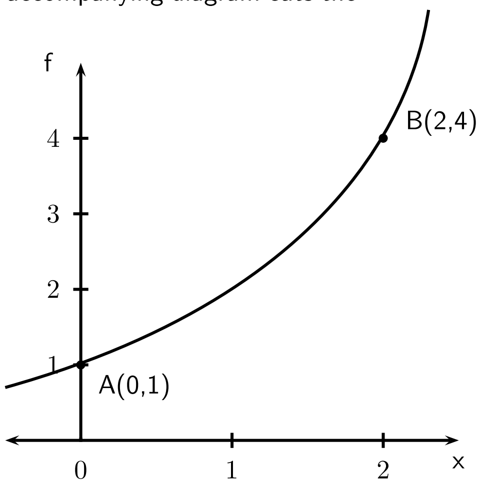


Figure 10.14: Graph of $g(x) = 3 \cdot 2^x + 2$.

?

Exercise: Exponential Functions and Graphs

1. Draw the graphs of $y = 2^x$ and $y = (\frac{1}{2})^x$ on the same set of axes.
   (a) Is the x-axis and asymptote or and axis of symmetry to both graphs ? Explain your answer.
   (b) Which graph is represented by the equation $y = 2^{-x}$? Explain your answer.
   (c) Solve the equation $2^x = (\frac{1}{2})^x$ graphically and check that your answer is correct by using substitution.
   (d) Predict how the graph $y = 2.2^x$ will compare to $y = 2^x$ and then draw the graph of $y = 2.2^x$ on the same set of axes.
2. The curve of the exponential function f in the accompanying diagram cuts the



y-axis at the point A(0; 1) and B(2; 4) is on f.
   (a) Determine the equation of the function f.
   (b) Determine the equation of h, the function of which the curve is the reflection of the curve of f in the x-axis.
   (c) Determine the range of h.

10.13 End of Chapter Exercises

1. Given the functions $f(x) = -2x^2 - 18$ and $g(x) = -2x + 6$
   (a) Draw f and g on the same set of axes.
   (b) Calculate the points of intersection of f and g.
   (c) Hence use your graphs and the points of intersection to solve for x when:
      i. $f(x) > 0$
      ii. $\frac{f(x)}{g(x)} \leq 0$
   (d) Give the equation of the reflection of f in the x-axis.
2. After a ball is dropped, the rebound height of each bounce decreases. The equation $y = 5(0.8)^x$ shows the relationship between x, the number of bounces, and y, the height of the bounce, for a certain ball. What is the approximate height of the fifth bounce of this ball to the nearest tenth of a unit ?
3. Marc had 15 coins in five rand and two rand pieces. He had 3 more R2-coins than R5-coins. He wrote a system of equations to represent this situation, letting x represent the number of five rand coins and y represent the number of two rand coins. Then he solved the system by graphing.

(a) Write down the system of equations.

(b) Draw their graphs on the same set of axes.

(c) What is the solution?

Appendix A

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