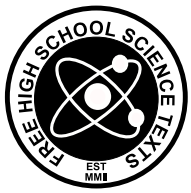

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The Free High School Science Texts: Textbooks for High School Students Studying the Sciences Mathematics Grades 10 - 12

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September 17, 2008

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Contents

Chapter 3

Rational Numbers - Grade 10

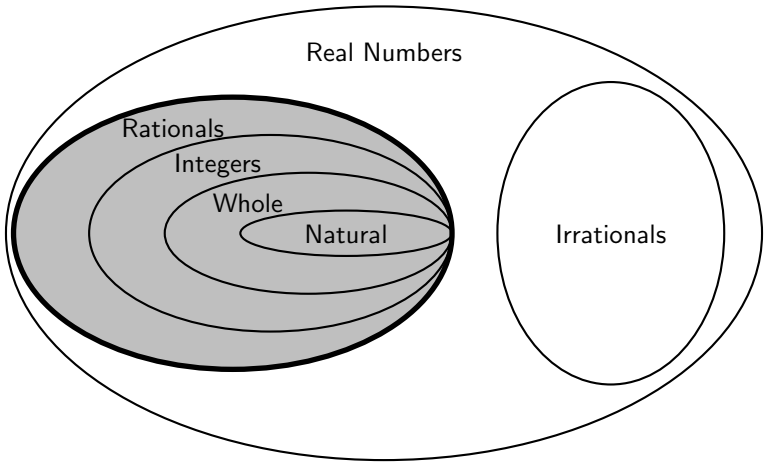
3.1 Introduction

As described in Chapter 2, a number is a way of representing quantity. The numbers that will be used in high school are all real numbers, but there are many different ways of writing any single real number.

This chapter describes *rational numbers*.

3.2 The Big Picture of Numbers

All numbers inside the grey oval are rational numbers.



3.3 Definition

The following numbers are all rational numbers.

$$\frac{10}{1}, \quad \frac{21}{7}, \quad \frac{-1}{-3}, \quad \frac{10}{20}, \quad \frac{-3}{6} \tag{3.1}$$

You can see that all the denominators and all the numerators are integers[1].

Definition: Rational Number
A rational number is any number which can be written as:

$$\frac{a}{b} \tag{3.2}$$

where a and b are integers and $b \neq 0$.

[1] Integers are the counting numbers (1, 2, 3, ...), their opposites (-1, -2, -3, ...), and 0.

Important: Only fractions which have a numerator and a denominator that are integers are rational numbers.

This means that all integers are rational numbers, because they can be written with a denominator of 1.

Therefore, while

$$\frac{\sqrt{2}}{7}, \quad \frac{-1{,}33}{-3}, \quad \frac{\pi}{20}, \quad \frac{-3}{6{,}39} \tag{3.3}$$

are **not examples** of rational numbers, because in each case, either the numerator or the denominator is not an integer.

Exercise: Rational Numbers

1. If a is an integer, b is an integer and c is not an integer, which of the following are rational numbers:

   (a) $\frac{5}{6}$ (b) $\frac{a}{3}$ (c) $\frac{b}{2}$ (d) $\frac{1}{c}$

2. If $\frac{a}{1}$ is a rational number, which of the following are valid values for a?

   (a) 1 (b) -10 (c) $\sqrt{2}$ (d) $2{,}1$

3.4 Forms of Rational Numbers

All integers and fractions with integer numerators and denominators are rational numbers. There are two more forms of rational numbers.

Activity :: Investigation : Decimal Numbers

You can write the rational number $\frac{1}{2}$ as the decimal number 0,5. Write the following numbers as decimals:

1. $\frac{1}{4}$
2. $\frac{1}{10}$
3. $\frac{2}{5}$
4. $\frac{1}{100}$
5. $\frac{2}{3}$

Do the numbers after the decimal comma end or do they continue? If they continue, is there a repeating pattern to the numbers?

You can write a rational number as a decimal number. Therefore, you should be able to write a decimal number as a rational number. Two types of decimal numbers can be written as rational numbers:

1. decimal numbers that end or *terminate*, for example the fraction $\frac{4}{10}$ can be written as 0,4.

2. decimal numbers that have a repeating pattern of numbers, for example the fraction $\frac{1}{3}$ can be written as $0{,}3333\overline{3}$.

For example, the rational number $\frac{5}{6}$ can be written in decimal notation as $0{,}8333\overline{3}$, and similarly, the decimal number 0,25 can be written as a rational number as $\frac{1}{4}$.

Important: Notation for Repeating Decimals

You can use a bar over the repeated numbers to indicate that the decimal is a repeating decimal.

3.5 Converting Terminating Decimals into Rational Numbers

A decimal number has an integer part and a fractional part. For example, $10{,}589$ has an integer part of 10 and a fractional part of $0{,}589$ because $10 + 0{,}589 = 10{,}589$. The fractional part can be written as a rational number, i.e. with a numerator and a denominator that are integers.

Each digit after the decimal point is a fraction with denominator in increasing powers of ten. For example:

- $\frac{1}{10}$ is $0{,}1$
- $\frac{1}{100}$ is $0{,}01$

This means that:

$$\begin{aligned} 2{,}103 &= 2 + \frac{1}{10} + \frac{0}{100} + \frac{3}{1000} \\ &= 2\frac{103}{1000} \\ &= \frac{2103}{1000} \end{aligned}$$

Exercise: Fractions

1. Write the following as fractions:

   (a) $0{,}1$ (b) $0{,}12$ (c) $0{,}58$ (d) $0{,}2589$

3.6 Converting Repeating Decimals into Rational Numbers

When the decimal is a repeating decimal, a bit more work is needed to write the fractional part of the decimal number as a fraction. We will explain by means of an example.

If we wish to write $0{,}\overline{3}$ in the form $\frac{a}{b}$ (where a and b are integers) then we would proceed as follows

$$\begin{aligned} x &= 0{,}33333\ldots && && (3.4) \\ 10x &= 3{,}33333\ldots && \text{multiply by 10 on both sides} && (3.5) \\ 9x &= 3 && \text{subtracting (3.4) from (3.5)} \\ x &= \frac{3}{9} = \frac{1}{3} \end{aligned}$$

And another example would be to write $5,\overline{432}$ as a rational fraction

$$x = 5,432432432\ldots \quad (3.6)$$
$$1000x = 5432,432432432\ldots \quad \text{multiply by 1000 on both sides} \quad (3.7)$$

$$999x = 5427 \quad \text{subtracting (3.6) from (3.7)}$$
$$x = \frac{5427}{999} = \frac{201}{37}$$

For the first example, the decimal number was multiplied by 10 and for the second example, the decimal number was multiplied by 1000. This is because for the first example there was only one number (i.e. 3) that recurred, while for the second example there were three numbers (i.e. 432) that recurred.

In general, if you have one number recurring, then multiply by 10, if you have two numbers recurring, then multiply by 100, if you have three numbers recurring, then multiply by 1000. Can you spot the pattern yet?

The number of zeros after the 1 is the same as the number of recurring numbers.

But not all decimal numbers can be written as rational numbers, because some decimal numbers like $\sqrt{2} = 1,4142135...$ is an irrational number and cannot be written with an integer numerator and an integer denominator. However, when possible, you should always use rational numbers or fractions instead of decimals.

?

Exercise: Repeated Decimal Notation

1. Write the following using the repeated decimal notation:
   (a) $0,11111111\ldots$
   (b) $0,1212121212\ldots$
   (c) $0,123123123123\ldots$
   (d) $0,11414541454145\ldots$
2. Write the following in decimal form, using the repeated decimal notation:
   (a) $\frac{2}{3}$
   (b) $1\frac{3}{11}$
   (c) $4\frac{5}{6}$
   (d) $2\frac{1}{9}\ldots$
3. Write the following decimals in fractional form:
   (a) $0,6333\ldots$
   (b) $5,3131\overline{31}$
   (c) $11,570571\ldots$
   (d) $0,999999\ldots$

3.7 Summary

The following are rational numbers:

- Fractions with both denominator and numerator as integers.
- Integers.
- Decimal numbers that end.
- Decimal numbers that repeat.

3.8 End of Chapter Exercises

1. If a is an integer, b is an integer and c is not an integer, which of the following are rational numbers:
   (a) $\frac{5}{6}$
   (b) $\frac{a}{3}$
   (c) $\frac{b}{2}$
   (d) $\frac{1}{c}$
2. Write each decimal as a simple fraction:
   (a) $0{,}5$
   (b) $0{,}12$
   (c) $0{,}6$
   (d) $1{,}59$
   (e) $12{,}27\overline{7}$
3. Show that the decimal $3{,}2\dot{1}\dot{8}$ is a rational number.
4. Showing all working, express $0{,}7\dot{8}$ as a fraction $\frac{a}{b}$ where $a, b \in Z$.

Appendix A

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